THE VOTER MODEL IN A RANDOM ENVIRONMENT IN $\mathbb{Z}^d$

DAYUE CHEN

The voter model is an interacting particle system, describing the collective behavior of voters who constantly update their political positions. In this talk, voters are represented by vertices of the Euclidean lattice $\mathbb{Z}^d$. The voter at x may hold either of two political positions, denoted by 0 or 1. Let $\eta(x)$ be the political position of voter x and the collection $\eta = \{\eta(x); x \in \mathbb{Z}^d\}$ be an element of $\{0,1\}^{\mathbb{Z}^d}$. The voter at x updates his political position at a random time, following the exponential distribution with parameter $\sum_z \mu_{xz}$, where the summation is over $2d$ nearest neighbors. At the time of update the voter takes the position of his neighbor y with probability $\mu_{xy}/(\sum_z \mu_{xz})$. When $\mu_e \equiv 1$, this is a model well studied in Chapter V of [9].

The voter model can be constructed by the graphical representation, see §3.6 of [9]. This approach not only works for all positive μ_{xy}, but also clearly exhibits the duality relation which will be used in our proof.

We are interested in the case when $(\mu_e, e \in E_d)$ are i.i.d. random variables satisfying $\mu_e \geq 1$, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. There have been very few literatures about the voter model in random environments. As far as we know, only the one dimensional voter model in a random environment has been explored in [7]. As usual, one would like to identify all invariant measures. When all voters take the same position, there will be no change thereafter. Therefore the configurations that $\eta(x) \equiv 0$ or 1 are traps of the voter model and the measures δ_0 and δ_1 of point mass are invariant. With an extra effort, we are able to identify all invariant measures.

Theorem 0.1. *Let $d = 1$ or 2. Suppose that (μ_e) are i.i.d. and $\mu_e \geq 1$ $\mathbb{P}$-a.s. There exists $\Omega_0 \subseteq \Omega$ with $\mathbb{P}(\Omega_0) = 1$. For any $\omega \in \Omega_0$, the voter model has only two extremal invariant measures: δ_0 and δ_1.*

This extends Corollary 1.13 of Chapter V of [9]. In light of Example 1.5 of Chapter V of [9], the conclusion of the theorem generally does not hold when $d \geq 3$.

The proof of the theorem involves the duality and the dual of the voter model is a coalescing Markov chain taking values on the set of all finite set of vertices of $\mathbb{Z}^d$. When the initial state is a singleton, the coalescing Markov chain always takes value on singletons. If we identify singleton $\{x\}$ with vertex x, then the coalescing Markov chain is exactly a continuous-time random walk in a random environment. Intuitively, a walker stays at x for an exponential time with parameter μ_x, jumps to a nearest neighbor, say y, with probability $\mu_{xy}/(\sum_z \mu_{xz})$. This is also called the *variable speed* *random walk* or the *random conductance model*. There is a large amount of literatures on random walks in random environments, e.g., [1], [5] and [6].

Let $\{X_t\}$ and $\{Y_t\}$ be two independent variable speed random walks. The problem we are interested is reduced to the property that $X_t = Y_t$ infinitely often. More specifically, we say $X_t = Y_t$ infinitely often if there exists an infinite sequence of random times $\{t_1, t_2, ...\}$ with $t_1 < t_2 < ...$ and $\lim_{i\to\infty} t_i = \infty$, such that $X(t_i) = Y(t_i)$ for all $i \geq 1$.

Theorem 0.2. *Let $d = 2$. Suppose that $(\mu_e, e \in E_d)$ are i.i.d. and $\mu_e \geq 1$ $\mathbb{P}$-a.s. There exists $\Omega_0 \subseteq \Omega$ with $\mathbb{P}(\Omega_0) = 1$. Let $\omega \in \Omega_0$ and $\mathbb{P}_\omega$ denote the probability conditional on the environment. If $\{X_t\}$ and $\{Y_t\}$ are two independent variable speed random walks starting from x and y respectively, then $\mathbb{P}_\omega(X_t = Y_t$ infinitely often$) = 1$.*

The proof follows a similar idea which is used in [3] and the key to the proof is the heat kernel estimates obtained by Barlow and Deuschel [1]. For more references related to collisions of two random walks, we refer readers to [2] [4] and [8]. Again the conclusion of the theorem holds trivially in dimension one, and fails in general when the dimension is 3 or greater.

This talk is based on my joint paper with Zhichao Shan.

References

[1] M. T. Barlow and J.-D. Deuschel, *Invariance principle for the random conductance model with unbounded conductances*, The Annals of Probability **38** (2010), no. 1, 234–276.

[2] M. T. Barlow, Y. Peres, and P. Sousi, *Collisions of random walks* (2010). preprint, available at `http://arxiv.org/abs/1003.3255`.

[3] X. Chen and D. Chen, *Two random walks on the open cluster of $\mathbb{Z}^2$ meet infinitely often*, Science China Mathematics **53** (2010), no. 8, 1971–1978.

[4] ______, *Some sufficient conditions for infinite collisions of simple random walks on a wedge comb*, Electronic Journal of Probability **16** (2011), 1341–1355.

[5] T. Delmotte, *Parabolic Harnack inequality and estimates of Markov chains on graphs*, Rev. Mat. Iberoamericana **15** (1999), 181–232.

[6] T. Delmotte and J.-D. Deuschel, *On estimating the derivatives of symmetric diffusions in stationary random environment, with applications to $\nabla\phi$ interface model*, Probab. Theory and Related Fields **133** (2005), 358–390.

[7] I. Ferreira, *The probability of survival for the biased voter model in a random environment*, Stochastic Processes and Their Applications **34** (1990), 25–38.

[8] M. Krishnapur and Y. Peres, *Recurrent graphs where two independent random walks collide finitely often*, Elect. Comm. Probab. **9** (2004), 72–81.

[9] T. M. Liggett, *Interacting Particle Systems*, Springer-Verlag, New York, 1985.

School of Mathematical Sciences, Peking University, Beijing, 100871, China.
E-mail address: `dayue@pku.edu.cn`