# Hyperbolic Scaling Limits in a Regime of Shock Waves A Synthesis of Probabilistic and PDE Techniques

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# Historical Notes (Hyperbolic systems)

**C.Morrey** (1955): Idea of scaling limits for mechanical models.

**R.L.Dobrushin** + coworkers (1980–85): One-dimensional hard rods and harmonic oscillators. Continuum of conservation laws.

**H.Rost** (1981): Asymmetric exclusion $\rightarrow$ rarefaction waves.

**F.Rezakhanlou** (1991): Coupling techniques for general attractive systems in a regime of shock waves. Single conservation laws only.

**H.-T. Yau** (1991) + Olla - Varadhan - Yau (1993): Preservation of local equilibrium in a smooth regime via the method of relative entropy. Hamiltonian dynamics with conservative, diffusive noise.

**JF** (2001–): Stochastic theory of compensated compactness. Further results with B. Tóth, Kati Nagy and C. Bahadoran. Shocks, non - attractive models, couples of conservation laws.

# Models and Methods

- The Anharmonic Chain with Conservative Noise. Physical and Artificial Viscosity. Ginzburg - Landau perturbation.

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- Stochastic Theory of Compensated Compactness
- Interacting Exclusions with Creation and Annihilation: Relaxation Scheme Replaces the missing LSI.

# Hyperbolic Systems of Conservation Laws

- $t \geq 0$, $x \in \mathbb{R}$, $u = u(t,x)$, $u, \Phi(u) \in \mathbb{R}^d$ :

$$\partial_t u(t,x) + \partial_x \Phi(u(t,x)) = 0\,;$$

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- Entropy Production:

$$X(h,u) := \partial_t h(u) + \partial_x J(u) \approx 0\,??$$

  beyond shocks in the sense of distributions.

# The Vanishing Viscosity Limit

- Parabolic Approximation:

$$\partial_t u_\sigma(t,x) + \partial_x \Phi(u_\sigma(t,x)) = \sigma \partial_x^2 u_\sigma(t,x)\,;$$

$u_\sigma \to u$ as $0 < \sigma \to 0\,??$

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- A Priori Bound for Entropy Production:

$$X(h,u) = \sigma \partial_x (h'(u) \cdot \partial_x u) - \sigma(\partial_x u \cdot h''(u) \partial_x u)$$

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- The bound does not vanish!! WE DO NOT HAVE ANY STRONG COMPACTNESS ARGUMENT!!

# Compensated Compactness

- Young Measure: $d\Theta := dt\,dx\,\theta_{t,x}(dy)$ represents $u$ if $\theta_{t,x}$ is the Dirac mass at $u(t,x)$. Hence $u_\sigma$ is relative compact in a space of measures.

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- F. Murat: Sending $\sigma \to 0$, the above decomposition implies

$$\theta_{t,x}(h_1 J_2) - \theta_{t,x}(h_2 J_1) = \theta_{t,x}(h_1)\theta_{t,x}(J_2) - \theta_{t,x}(h_2)\theta_{t,x}(J_1)$$

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  for all couples of entropy pairs.
- L. Tartar - R. DiPerna: The limiting $\theta$ is Dirac, therefore it represents a weak solution.

# The Anharmonic Chain

- Configurations: $\omega = \{(p_k, r_k) : k \in \mathbb{Z}\}$ , $p_k, r_k \in \mathbb{R}$ are the momentum and the deformation at site $k \in \mathbb{Z}$ . Dynamics:

$$\dot{p}_k = V'(r_k) - V'(r_{k-1}) \quad \text{and} \quad \dot{r}_k = p_{k+1} - p_k \,,$$

$V(x) \approx x^2/2$ at infinity, sub - exponential growth of $p_k, r_k$ .
Generator: the Liouville operator $\mathcal{L}_0$ , $\partial_t \varphi(\omega(t)) = \mathcal{L}_0 \varphi(\omega)$ .
Hyperbolic scaling: $\pi_\varepsilon(t,x) := p_k(t/\varepsilon)$ , $\rho_\varepsilon(t,x) := r_k(t/\varepsilon)$ if $|k\varepsilon - x| < \varepsilon/2$ , as $0 < \varepsilon \to 0$ .

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- Classical conservation laws: $p_k$ , $r_k$ and $H_k := p_k^2/2 + V(r_k)$ ;
  $\partial_t H_k = p_{k+1} V'(r_k) - p_k V'(r_{k-1})$ .
  Is there any other??
  Stationary product measures: $\lambda_{\beta,\pi,\gamma}$ , $p_k \sim N(\pi, 1/\beta)$ ,
  Lebesgue density of $r_k \sim e^{\gamma x - \beta V(x)}$ .
  HDL: Compressible Euler equations?
  Strong ergodic hypothesis: Description of all stationary states and conservation laws!!

# Physical Viscosity

- The anharmonic chain can be regularized by adding stochastic perturbations to the equations of momenta.
  Random exchange of momenta: $\mathcal{L} = \mathcal{L}_0 + \sigma \mathcal{S}_{ep}$ ,

$$\mathcal{S}_{ep}\varphi(\omega) = \sum_{k\in\mathbb{Z}} (\varphi(\omega^{k,k+1}) - \varphi(\omega)) \,,$$

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  $\omega \to \omega^{k,k+1}$ means $p_k \leftrightarrow p_{k+1}$ .
- The classical conservation laws are OK, and the strong ergodic hypothesis (F - Funaki - Lebowitz 1994) implies the triplet of compressible Euler equations in a smooth regime with periodic boundary conditions. The asymptotic preservation of local equilibrium follows by the relative entropy argument.

- **Ginzburg - Landau perturbation.** Stochastic dynamics:

$$
\begin{aligned}
dp_k &= \left(V'(r_k) - V'(r_{k-1})\right) dt + \sigma \left(p_{k+1} + p_{k-1} - 2p_k\right) dt \\
&\quad + \sqrt{2\sigma}\left(dw_k - dw_{k-1}\right), \quad dr_k = \left(p_{k+1} - p_k\right) dt\,, \quad k \in \mathbb{Z}\,,
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- Energy is not conserved, $\lambda_{\pi,\gamma} := \lambda_{1,\pi,\gamma}$ are stationary. We have convergence to classical solutions of the nonlinear sound equation of elastodynamics:

$$\partial_t \pi = \partial_x S'(\rho) \quad \text{and} \quad \partial_t \rho = \partial_x \pi$$

  as $\int V'(r_k)\,d\lambda_{\pi,\gamma} = \gamma = S'(\rho)$ if $\int r_k\,d\lambda_{\pi,\gamma} = \rho = F'(\gamma)\,,$

$$S(\rho) := \sup_{\gamma}\{\gamma\rho - F(\gamma)\}\,,\ F(\gamma) := \log \int_{-\infty}^{\infty} \exp(\gamma x - V(x))\,dx\,.$$

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- Conservation laws and stationary states as before: Again the sound equation is expected as the result of the hyperbolic scaling limit.

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- A technical condition of asymptotic normality is also needed: $V''(x)$ converges at an exponential rate as $x \to \pm\infty$.
- Our only hypothesis on the initial distribution is the entropy bound: $S[\mu_{0,\varepsilon,n}|\lambda_{0,0}] = O(n)$, where $\mu_{t,\varepsilon,n}$ denotes the joint distribution of the variables $\{(p_k(t), r_k(t)) : |k| \leq n\}$, and $S[\mu|\lambda] := \int \log f \, d\mu$, $f = d\mu/d\lambda$.

- **Main Result.** The distributions $\mathsf{P}_\varepsilon$ of the empirical process $(\pi_\varepsilon, \rho_\varepsilon)$ form a tight family with respect to the weak topology of the $C$ space of trajectories, and its limit distributions are all concentrated on a set of weak solutions to the sound equation.

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- The notion of weak convergence above changes from step to step of the argument. We start with the Young measure of the block - averaged empirical process $(\hat{\pi}_\varepsilon, \hat{\rho}_\varepsilon)$, finally we get tightness in the strong local $L^p(\mathbb{R}^2_+)$ topology if $p < 2$.

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- Compensated compactness is the most relevant keyword of the proofs, results by J. Shearer (1994) and Serre - Shearer (1994) are applied at the end.
- In the case of systems the uniqueness of the hydrodynamic limit is still a formidable open problem, we are not able to prove the desired local bounds for our stochastic models.

# On the ideas of the proof

- **The Main Steps.** We follow the argumentation of the vanishing viscosity approach. There is a rich family of Lax entropy pairs $(h, J)$, entropy production $X_\varepsilon := \partial_t h(\hat{\pi}_\varepsilon, \hat{\rho}_\varepsilon) + \partial_x J(\hat{\pi}_\varepsilon, \hat{\rho}_\varepsilon)$ is considered as a generalized function.

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- First difficulty: to identify the macroscopic flux $J$ in the microscopic expression of $\mathcal{L}_0 h$, and to show that the remainders do vanish in the limit.
- Replace block averages of the microscopic currents of momenta with their equilibrium expectations via LSI. It is based on our a priori bounds on relative entropy and its Dirichlet form.
- Recover at the microscopic (mesoscopic) level the basic structure of the vanishing viscosity limit.
- Launch the stochastic theory of compensated compactness.

# The a Priori Bounds

- **The Entropy Bound and LSI.** The initial condition implies that

$$S[\mu_{t,\varepsilon,n}|\lambda_{0,0}] + \sigma(\varepsilon)\int_0^t D[\mu_{s,\varepsilon,n}|\lambda_{0,0}]\,ds \le C\,(t+n)$$

for all $t, n, \varepsilon$ with the same constant $C$ ; $f_n := d\mu_{t,\varepsilon,n}/d\lambda_{0,0}$ ,

$$D := \sum_{k=-n}^{n-1}\int(\nabla_1\partial_k\sqrt{f_n})^2\,d\lambda + \sum_{k=-n}^{n-1}\int(\nabla_1\tilde{\partial}_k\sqrt{f_n})^2\,d\lambda\,,$$

$\nabla_\ell\xi_k := (1/\ell)(\xi_{k+\ell}-\xi_k)$ , $\partial_k := \partial/\partial p_k$ and $\tilde{\partial}_k := \partial/\partial r_k$ .

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$\nabla_\ell\xi_k := (1/\ell)(\xi_{k+\ell}-\xi_k)$ , $\partial_k := \partial/\partial p_k$ and $\tilde{\partial}_k := \partial/\partial r_k$ .

- Since $V$ is convex, the following LSI holds true. Given $\bar{r}_{\ell,k} = v$ , let $\mu^\rho_{\ell,k}$ and $\lambda^\rho_{\ell,k}$ denote the conditional distributions of the variables $r_k, r_{k-1}, \ldots, r_{k-\ell+1}$ , and set $f^v_{\ell,k} := d\mu^\rho_{\ell,k}/d\lambda^\rho_{\ell,k}$ , then

$$\int \log f^\rho_{\ell,k}\,d\mu^\rho_{\ell,k} \le \ell^2 C_{\mathrm{lsi}} \sum_{j=k-\ell+1}^{k-1}\int\left(\nabla_1\tilde{\partial}_k(f^\rho_{\ell,k})^{1/2}\right)^2\,d\lambda^\rho_{\ell,k}\,.$$

- **Replacement of the microscopic flux.** Combining LSI and the entropy inequality $\int \varphi \, d\mu \leq S[\mu|\lambda] + \log \int e^{\varphi} \, d\lambda$ we get

$$\sum_{|k|<n} \int_0^t \int \left(\bar{V}'_{\ell,k} - S'(\bar{r}_{\ell,k})\right)^2 \, d\mu_{s,\varepsilon} \, ds \leq C_1 \left(\frac{nt}{\ell} + \frac{n\ell^2}{\sigma(\varepsilon)}\right) ,$$

where $\bar{\xi}_{\ell,k} := (\xi_k + \xi_{k-1} + \cdots + \xi_{k-\ell+1})/\ell$, e.g. $V'_k = V'(r_k)$. Similar bounds control the differences $\bar{r}_{\ell,k+\ell} - \bar{r}_{\ell,k}$ and $\hat{r}_{\ell,k} - \bar{r}_{\ell,k}$.

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- Entropy production $X_\varepsilon$ is written in terms of the "mollified" block averages $\hat{\xi}_{\ell,k}$ , these are defined by means of a triangular weight function. Mesoscopic blocks of size $\ell = \ell(\varepsilon)$ are used:

$$\lim_{\varepsilon \to 0} \frac{\ell(\varepsilon)}{\sigma(\varepsilon)} = 0 \quad \text{and} \quad \lim_{\varepsilon \to 0} \frac{\varepsilon \ell^3(\varepsilon)}{\sigma(\varepsilon)} = +\infty \, .$$

# Lax Entropy Pairs

- One critical term of $X_\varepsilon$ can be computed as

$$\begin{aligned} X_{0,k} &:= \mathcal{L}_0 h(\hat{p}_{\ell,k},\hat{r}_{\ell,k}) + J(\hat{p}_{\ell,k+1},\hat{r}_{\ell,k+1}) - J(\hat{p}_{\ell,k},\hat{r}_{\ell,k}) \\ &\approx h'_u(\hat{p}_{\ell,k},\hat{r}_{\ell,k})(\hat{V}'_{\ell,k} - \hat{V}'_{\ell,k-1}) + h'_v(\hat{p}_{\ell,k},\hat{r}_{\ell,k})(\hat{p}_{\ell,k+1} - \hat{p}_{\ell,k}) \\ &+ J'_u(\hat{p}_{\ell,k},\hat{r}_{\ell,k})(\hat{p}_{\ell,k+1} - \hat{p}_{\ell,k}) + J'_v(\hat{p}_{\ell,k},\hat{r}_{\ell,k})(\hat{r}_{\ell,k+1} - \hat{r}_{\ell,k}) \,. \end{aligned}$$

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&+ J'_u(\hat{p}_{\ell,k}, \hat{r}_{\ell,k})(\hat{p}_{\ell,k+1} - \hat{p}_{\ell,k}) + J'_v(\hat{p}_{\ell,k}, \hat{r}_{\ell,k})(\hat{r}_{\ell,k+1} - \hat{r}_{\ell,k}) \,.
\end{aligned}
$$

- Since $h'_\pi(\pi,\rho)S''(\rho) + J'_\rho(\pi,\rho) = h'_\rho(\pi,\rho) + J'_\pi(\pi,\rho) = 0\,,$

$$
X_{0,k} \approx h'_u(\hat{\pi}_{\ell,k}, \hat{\rho}_{l,k})\,(\hat{V}'_{\ell,k} - \hat{V}'_{\ell,k-1} - S''(\hat{r}_{\ell,k})(\hat{r}_{\ell,k+1} - \hat{r}_{\ell,k}))\,.
$$

Observe now that $\hat{\xi}_{\ell,k+1} - \hat{\xi}_{\ell,k} = (1/\ell)(\bar{\xi}_{\ell,k+\ell} - \bar{\xi}_{\ell,k})$, thus the substitution $\bar{V}'_{\ell,k} \approx S'(\bar{r}_{\ell,k})$ results in $X_{0,k} \approx 1/\ell$.
In fact $(\varepsilon\ell(\varepsilon)\sigma(\varepsilon))^{-1}$ is the order of the replacement error; that is why we need?? $\varepsilon\sigma^2(\varepsilon) \to +\infty$ and the sharp explicit bounds provided by the logarithmic Sobolev inequality.

# Stochastic Compensated Compactness

For Lax entropy pairs $(h, J)$ set

$$X_\varepsilon(\psi, h) := -\int_0^\infty \int_{-\infty}^\infty \big(h(\hat{u}_\varepsilon)\psi_t'(t,x) + J(\hat{u}_\varepsilon)\psi_x'(t,x)\big)\; dx\, dt\,,$$

where $u_\varepsilon = (\pi, \rho)$, and the test function $\psi$ is compactly supported in the interior of $\mathbb{R}^2_+$. Another test function $\phi$ localizes $X$. An entropy pair $(h, J)$ is well controlled if $X_\varepsilon$ decomposes as $X_\varepsilon(\psi, h) = Y_\varepsilon(\psi, h) + Z_\varepsilon(\psi, h)$, and we have two random functionals $A_\varepsilon(\phi, h)$ and $B_\varepsilon(\phi, h)$ such that

$$|Y_\varepsilon(\psi\phi, h)| \le A_\varepsilon(\phi, h)\|\psi\|_+ \quad \text{and} \quad |Z_\varepsilon(\psi, h)| \le B_\varepsilon(\phi, h)\|\psi\| \;:$$

$\lim \mathrm{E} A_\varepsilon(\phi, h) = 0$ and $\limsup \mathrm{E} B_\varepsilon(\phi, h) < +\infty$ as $\varepsilon \to 0$.

- The Young family $\Theta$ is defined as $d\Theta_\varepsilon := dt\, dx\, \theta_{\varepsilon,t,x}(du)$, where $\theta_{\varepsilon,t,x}$ is the Dirac mass at the actual value of $\hat{u}_\varepsilon$.

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- If $(h_1, J_1)$ and $(h_2, J_2)$ are well controlled entropy pairs with bounded second derivatives then the Div - Curl lemma holds true:

$$\theta_{t,x}(h_1 J_2) - \theta_{t,x}(h_2 J_1) = \theta_{t,x}(h_1)\theta_{t,x}(J_2) - \theta_{t,x}(h_2)\theta_{t,x}(J_1)$$

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- Now we are in a position to refer to the papers by J. Shearer and Serre - Shearer on an $L^p$ theory of compensated compactness. The Dirac property of the Young measure follows by means of their moderately increasing entropy families.

# Interacting Exclusions, Tóth - Valkó (2002)

- Charged particles: $\omega = (\omega_k = 0, \pm 1 : k \in \mathbb{Z})$, $\eta_k := \omega_k^2$.

$$\mathcal{L}_0 \varphi(\omega) = \frac{1}{2} \sum_{k \in \mathbb{Z}} (\eta_k + \eta_{k+1} + \omega_k - \omega_{k+1})(\varphi(\omega^{k,k+1}) - \varphi(\omega)) .$$

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- Conservation laws: $\mathcal{L}_0 \omega_k = \mathrm{j}^{\omega}_{k-1} - \mathrm{j}^{\omega}_{k}$ and $\mathcal{L}_0 \eta = \mathrm{j}^{\eta}_{k-1} - \mathrm{j}^{\eta}_{k}$, where

$$\begin{aligned} \mathrm{j}^{\omega}_{k} :&= (1/2)\left(\eta_k + \eta_{k+1} - 2\omega_k \omega_{k+1} + \omega_k \eta_{k+1} - \eta_k \omega_{k+1}\right) \\ &\quad + (1/2)(\eta_k - \eta_{k+1}), \\ \mathrm{j}^{\eta}_{k} :&= (1/2)\left(\omega_k + \omega_{k+1} - \omega_k \eta_{k+1} - \eta_k \omega_{k+1} + \eta_k - \eta_{k+1}\right). \end{aligned}$$

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- All Bernoulli measures $\lambda_{u,\rho}$ are stationary: $\int \omega_k \, d\lambda_{u,\rho} = u$ and $\int \eta_k \, d\lambda_{u,\rho} = \rho$.
  $\int \mathrm{j}_k^{\omega} \, d\lambda_{u,\rho} = \rho - u^2$ and $\int \mathrm{j}_k^{\eta} \, d\lambda_{u,\rho} = u - u\rho$.

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  $\int \mathrm{j}^\omega_k\, d\lambda_{u,\rho} = \rho - u^2$ and $\int \mathrm{j}^\eta_k\, d\lambda_{u,\rho} = u - u\rho\,.$
- The hyperbolic scaling limit for $u \sim \bar{\omega}_{\ell,k}$ and $\rho \sim \bar{\eta}_{\ell,k}$ yields:

$$\partial_t u + \partial_x(\rho - u^2) = 0\,, \quad \partial_t\rho + \partial_x(u - u\rho) = 0 \quad \text{(Leroux)}$$

F - Tóth (2004).

# Creation and Annihilation

The action $\omega \to \omega^{k+}$ of creation means that $(\omega_k, \omega_{k+1}) \to (1, -1)$ if $\omega_k = \omega_{k+1} = 0$, while annihilation $\omega \to \omega^{k-}$ is defined by $(\omega_k, \omega_{k+1}) \to (0, 0)$ if $\omega_k = 1$ and $\omega_{k+1} = -1$. The generator of the composed process reads as $\mathcal{L}^* = \mathcal{L}_0 + \beta\, \mathcal{G}^*$, where $\beta > 0$ and

$$\begin{aligned} \mathcal{G}^*\varphi(\omega) := \sum_{k\in\mathbb{Z}} (1-\eta_k)(1-\eta_{k+1})(\varphi(\omega^{k+}) - \varphi(\omega)) \\ + \frac{1}{4} \sum_{k\in\mathbb{Z}} (\eta_k + \omega_k)(\eta_{k+1} - \omega_{k+1})(\varphi(\omega^{k-}) - \varphi(\omega)) . \end{aligned}$$

Creation - annihilation violates the conservation of particle number. The product measure $\lambda_{u,\rho}$ will be stationary if $\lambda_{u,\rho}[\omega_k = 0, \omega_{k+1} = 0] = \lambda_{u,\rho}[\omega_k = 1, \omega_{k+1} = -1]$, whence

$$\rho = F(u) := (1/3)(4 - \sqrt{4 - 3u^2})$$

is the criterion of equilibrium because the second root:

$$\tilde{F}(u) := (1/3)(4 + \sqrt{4 - 3u^2}) \geq 5/3 > 1 .$$

# Substitution

- **Equilibrium Expectations.** $\lambda_u^* := \lambda_{u,F(u)}$ , $|u| < 1$ is the family of our stationary product measures:
$\int \omega_k \, d\lambda_u^* = u$ , $\int \eta_k \, d\lambda_u^* = F(u)$ and $\int \mathrm{j}_k^\omega \, d\lambda_u^* = F(u) - u^2$ .

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- $\mathcal{G}^* \omega_k = \mathrm{j}^{\omega *}_{k-1} - \mathrm{j}^{\omega *}_k$ is also a difference of currents:

  $$\mathrm{j}^{\omega *}_k(\omega) := (1/4)(\eta_k + \omega_k)(\eta_{k+1} - \omega_{k+1}) - (1 - \eta_k)(1 - \eta_{k+1}),$$

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$$\partial_t u(t, x) + \partial_x (F(u) - u^2) = 0 \qquad \text{(CreAnni)}$$

is the expected result of the hyperbolic scaling limit.

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is the expected result of the hyperbolic scaling limit.
- Simply substitute $\rho = F(u)$ into the first equation of the Leroux system. Hyperbolic scaling: $\mathcal{G}^*$ has no contribution. Navier - Stokes correction??

# Main Result

- Since we do not want to postulate the smoothness of the macroscopic solution, the basic process is regularized by an overall stirring $\mathcal{S}_e$, the full generator reads as $\mathcal{L} := \mathcal{L}^* + \sigma(\varepsilon)\,\mathcal{S}_e$, and the theory of compensated compactness is applied.

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- Assume that the initial distributions satisfy

$$\lim_{\varepsilon\to 0} \varepsilon \sum_{k\in\mathbb{Z}} \varphi(\varepsilon k)\omega_k(0) = \int_{-\infty}^{\infty} \psi(x) u_0(x)\, dx$$

in probability for all compactly supported $\varphi \in C(\mathbb{R})$.

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  in probability for all compactly supported $\varphi \in C(\mathbb{R})$.
- The artificial viscosity $\sigma(\varepsilon)$ and the size $\ell = \ell(\varepsilon)$ of the mesoscopic block averages are the same as before. Our empirical process is defined as $\hat{u}_\varepsilon(t,x) := \hat{\omega}_{\ell,k}(t/\varepsilon)$ if $|\varepsilon k - x| < \varepsilon/2$.

- With C. Bahadoran and K. Nagy (EJP 2011) we prove that

$$\lim_{\varepsilon \to 0} \mathrm{E} \int_0^\tau \int_{-r}^r |u(t,x) - \hat{u}_\varepsilon(t,x)| \, dx \, dt = 0$$

for all $r, \tau > 0$, where $u$ is the uniquely specified weak entropy solution to the CreAnni equation with initial value $u_0$.

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- The coefficient $\beta > 0$ needs not be a constant, it is sufficient to assume that $\varepsilon\sigma^2(\varepsilon)\beta^{-4}(\varepsilon) \to +\infty$ and $\sigma(\varepsilon)\beta(\varepsilon) \to +\infty$ as $\varepsilon \to 0$.

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- The coefficient $\beta > 0$ needs not be a constant, it is sufficient to assume that $\varepsilon\sigma^2(\varepsilon)\beta^{-4}(\varepsilon) \to +\infty$ and $\sigma(\varepsilon)\beta(\varepsilon) \to +\infty$ as $\varepsilon \to 0$.
- The proof follows the standard technology of the stochastic theory of compensated compactness, the entropy production for entropy pairs $(h, J)$ of equation CreAnni has to be evaluated. However, the present logarithmic Sobolev inequality is not sufficient for the identification of $\partial_x J$ in the stochastic equation of $h$.

# Main Steps of the Proof

- **Entropy Production.** The local bound on relative entropy and an LSI involving $\mathcal{S}_e$ allow us to do the replacements $\bar{\mathrm{j}}_{\ell,k} \approx \mathfrak{J}(\bar{\omega}_{\ell,k}, \bar{\eta}_{\ell,k})$ :
  $\mathfrak{J} = \rho - u^2 \,, \quad \mathfrak{J} = u - u\rho \quad$ and $\quad \mathfrak{J} = C(u, \rho)$
  if $\mathrm{j} = \mathrm{j}^{\omega} \,, \quad \mathrm{j} = \mathrm{j}^{\eta} \quad$ and $\quad \mathrm{j} = \mathrm{j}^{\omega *}$ , respectively.
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  The explicit form of the bounds is the same as for $\bar{V}'$ above.
- Our entropy pairs $(h, J)$ satisfy $J'(u) = (F'(u) - 2u)h'(u)$ . Since $\mathcal{G}^*$ is reversible, one critical component of $X_\varepsilon$ reds as

  $$X^*_{0,k} := \mathcal{L}_0 h(\hat{\omega}_{\ell,k}) + J(\hat{\omega}_{\ell,k+1}) - J(\hat{\omega}_{\ell,k})$$
  $$\approx (1/\ell) h'(\hat{\omega}_{\ell,k}) \left(\bar{\eta}_{\ell,k} - \bar{\eta}_{\ell,k+\ell} + F'(\bar{\omega}_{\ell,k})(\bar{\omega}_{\ell,k+\ell} - \bar{\omega}_{\ell,k})\right) ,$$

  whence the required $X^*_{0,k} \approx 1/\ell$ would follow by $\bar{\eta}_{\ell,k} \approx F(\bar{\omega}_{\ell,k})$ . Since we do not have the appropriate logarithmic Sobolev inequality, another tool must be found.

# Relaxation in action

- $\eta_k$ appears with a negative sign in the formula of $\mathcal{G}^*\eta_k = -\mathrm{j}_{k-1}^{\omega *} - \mathrm{j}_k^{\omega *}$ and $\int \mathcal{G}^*\eta_k \, d\lambda_{u,\rho} = -2C(u,\rho)$, thus we hope to find a relaxation scheme. The approximate identities below reflect the underlying structure:

$$d\tilde{u}_\varepsilon + \partial_x(\tilde{\rho}_\varepsilon - \tilde{u}_\varepsilon^2)\,dt + \beta\,\partial_x C(\tilde{u}_\varepsilon, \tilde{\rho}_\varepsilon)\,dt \approx 0\,,$$
$$d\tilde{\rho}_\varepsilon + \partial_x(\tilde{u}_\varepsilon - \tilde{u}_\varepsilon\tilde{\rho}_\varepsilon) + (2\beta/\varepsilon)\,C(\tilde{u}_\varepsilon, \tilde{\rho}_\varepsilon)\,dt \approx 0\,,$$

where $\tilde{u}_\varepsilon \sim \bar{\omega}_{\ell,k}$ and $\tilde{\rho}_\varepsilon \sim \bar{\eta}_{\ell,k}$ by mollification.

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  where $\tilde{u}_\varepsilon \sim \bar{\omega}_{\ell,k}$ and $\tilde{\rho}_\varepsilon \sim \bar{\eta}_{\ell,k}$ by mollification.
- Since

$$(\rho - F(u))C(u,\rho) \geq \Psi(u,\rho) := (1/2)\,(\rho - F(u))^2\;,$$

  even the trivial Liapunov function $\Psi$ can be applied to conclude that $\bar{\eta}_{l,k} \approx F(\bar{\omega}_{l,k})$. This trick works well if $\varepsilon\sigma^2(\varepsilon)\beta^2(\varepsilon) \to +\infty$ as $\varepsilon \to 0$, a slightly better result can be proven by replacing $\Psi$ with a clever Lax entropy.

- **The End.** The Div - Curl lemma is now a consequence of our a priori bounds including $\bar{\eta}_{\ell,k} \approx F(\bar{\omega}_{\ell,k})$ . The uniqueness of the hydrodynamic limit follows by the Lax entropy inequality: $\limsup X_\varepsilon(\psi, h) \leq 0$ for $\psi \geq 0$ and convex $h$ . The bound on $Z_\varepsilon$ of the decomposition $X_\varepsilon = Y_\varepsilon + Z_\varepsilon$ does never vanish.

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- **Open problems:**
  Lax inequality for the anharmonic chain with artificial viscosity.
  Uniqueness of HDL to the Leroux system, say.
  Relaxation of $\varepsilon\sigma^2(\varepsilon) \to +\infty$ by a careful non - gradient analysis.
  Derivation of the compressible Euler equations with physical viscosity by adding energy and momentum preserving noise to the equations of the anharmonic chain.
  Navier - Stokes correction for creation and annihilation.