Localization for Brownian motion in a heavy tailed Poissonian potential

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Motivation

To understand the behavior of Brownian motion among randomly distributed (repulsive) obstacles.

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$\longrightarrow$ kill the Brownian motion by a random potential and condition to survive.

1. Setting

- $\left(\{B_t\}_{t\geq 0}, P_x\right)$: $\kappa\Delta$-Brownian motion on $\mathbb{R}^d$
- $\left(\omega = \sum_i \delta_{\omega_i}, \mathbb{P}\right)$: Poisson point process on $\mathbb{R}^d$ with unit intensity

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<u>Potential</u>

For a non-negative function v,

$$V_\omega(x) := \sum_i v(x - \omega_i).$$

(Typically $v(x) = 1_{B(0,1)}(x)$ or $|x|^{-\alpha} \wedge 1$ with $\alpha > d$.)

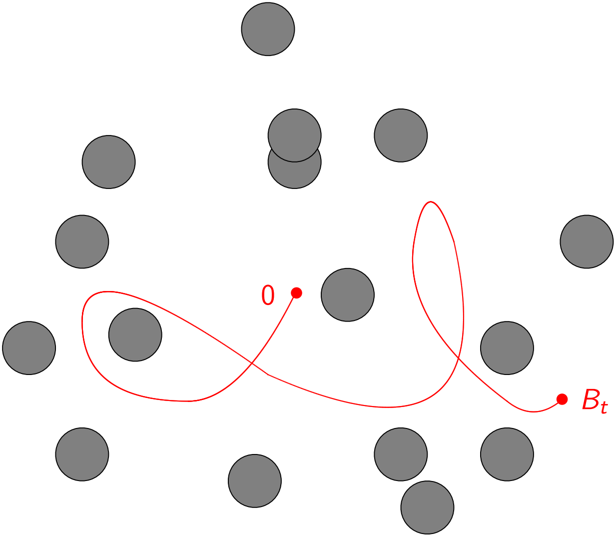
Annealed path measure

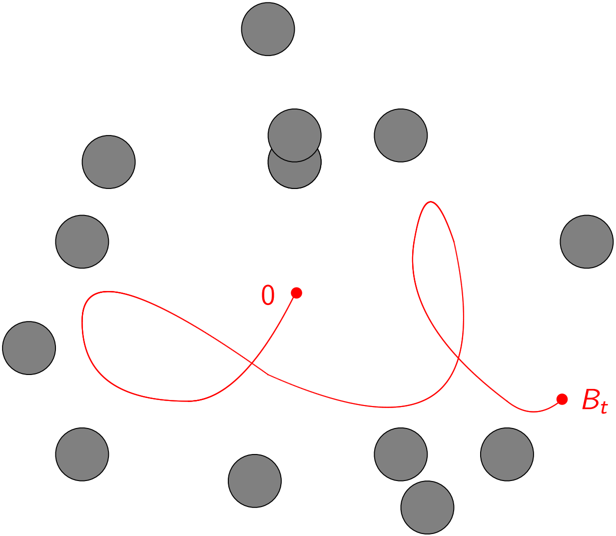
We are interested in the behavior of Brownian motion under the measure

$$Q_t(\,\cdot\,) = \frac{1}{Z_t}\exp\left\{-\int_0^t V_\omega(B_s)\mathrm{d}s\right\}\mathbb{P}\otimes P_0(\,\cdot\,),$$

$$Z_t = \mathbb{E}\otimes E_0\left[\exp\left\{-\int_0^t V_\omega(B_s)\mathrm{d}s\right\}\right].$$

The configuration is not fixed and hence Brownian motion and ω_i's try to avoid each other.


0
B_t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Brief history (only annealed)

- Anderson (1958): Localization of electron in random potentials.
- Donsker-Varadhan (1975): Asymptotics of Z_t in the case $v(x) = o(|x|^{-d-2})$.
- Pastur (1977): Asymptotics of Z_t in the case $v(x) \sim c|x|^{-\alpha}$ for $\alpha \in (d, d+2)$.
- Okura (1981): Asymptotics of Z_t in the case $v(x) \sim c|x|^{-d-2}$.
- Gärtner-Molchanov (1990, 1998): Localization of diffusion particle in general random potentials (weak sense).
- Sznitman (1991), Bolthausen (1994), Povel (1999): Strong localization of diffusion particle for compactly supported v.
- Asymptotics of Z_t in various settings in the name "parabolic Anderson model": Gärtner, Molchanov, König, Biskup, van der Hofstad, Mörters, Sidorova,...

2. Light tailed case

Donsker and Varadhan (1975)

When $v(x) = o(|x|^{-d-2})$ as $|x| \to \infty$,

$$\begin{aligned}
&\mathbb{E} \otimes E_0 \left[\exp\left\{ -\int_0^t V_\omega(B_s)\, ds \right\} \right] \\
&= \exp\left\{ -c(d) t^{\frac{d}{d+2}} (1 + o(1)) \right\} \\
&= P_0 \left(B_{[0,t]} \subset B(x, t^{\frac{1}{d+2}} R_0) \right) \mathbb{P} \left(\omega(B(x, t^{\frac{1}{d+2}} R_0)) = 0 \right),
\end{aligned}$$

as $t \to \infty$.

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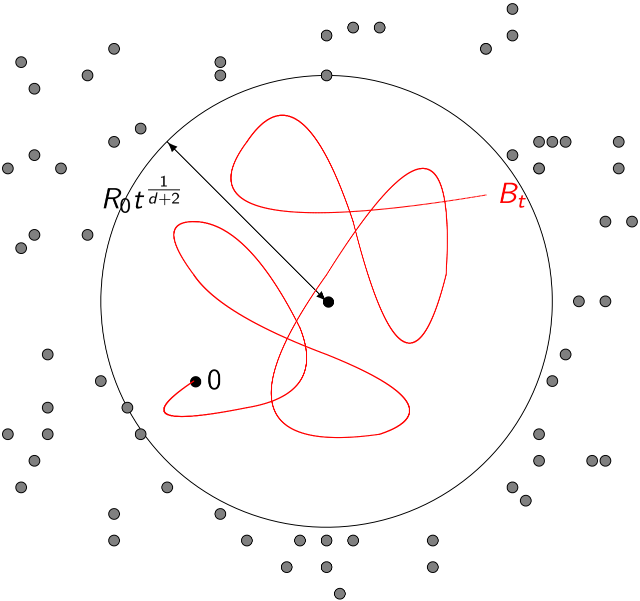
Remark

This is related to spectral asymptotics of $-\kappa\Delta + V_\omega$ (Lifshiz tail).


$R_0 t^{\frac{1}{d+2}}$
B_t
0

One specific strategy gives dominant contribution to the partition function.

$$\Downarrow$$

It occurs with high probability under the annealed path measure.

Sznitman (1991, $d = 2$) and Povel (1999, $d \geq 3$)

When v has a compact support, there exists

$$D_t(\omega) \in B\big(0, t^{\frac{1}{d+2}}(R_0 + o(1))\big)$$

such that

$$Q_t\left(B_{[0,t]} \subset B\big(D_t(\omega), t^{\frac{1}{d+2}}(R_0 + o(1))\big)\right) \xrightarrow{t\to\infty} 1.$$

Remark

Bolthausen (1994) proved the corresponding result for two-dimensional random walk model.

3. Heavy tailed case

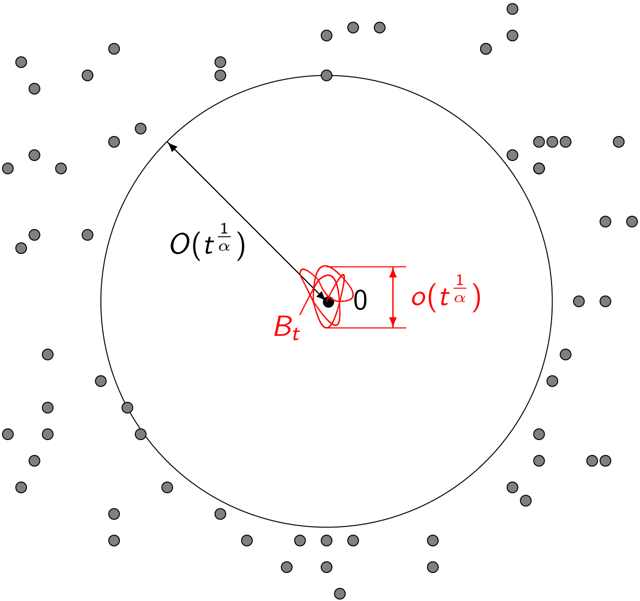
Pastur (1977)

When $v(x) \sim |x|^{-\alpha}$ as $|x| \to \infty$ with $\alpha \in (d, d+2)$,

$$\mathbb{E} \otimes E_0 \left[\exp \left\{ - \int_0^t V_\omega(B_s) \mathrm{d}s \right\} \right] = \exp \left\{ -a_1 t^{\frac{d}{\alpha}} \right\}$$

as $t \to \infty$, where

$$a_1 := |B(0,1)| \Gamma \left(\frac{\alpha - d}{\alpha} \right).$$


$O(t^{\frac{1}{\alpha}})$
0
$o(t^{\frac{1}{\alpha}})$
B_t

F. (2011)

When $v(x) = |x|^{-\alpha} \wedge 1$ with $\alpha \in (d, d+2)$,

$$\begin{aligned}\mathbb{E} \otimes E_0 \left[\exp\left\{ -\int_0^t V_\omega(B_s)\mathrm{d}s \right\} \right] \\ = \exp\left\{ -a_1 t^{\frac{d}{\alpha}} - (a_2 + o(1)) t^{\frac{\alpha+d-2}{2\alpha}} \right\}\end{aligned}$$

as $t \to \infty$, where

$$a_2 := \inf_{\|\phi\|_2 = 1} \left\{ \int \kappa |\nabla \phi(x)|^2 + C(d,\alpha)|x|^2 \phi(x)^2 \, dx \right\}.$$

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Remark

The proof relies on a general machinery developed by Gärtner-König (2000).

We believe that the 2nd term (in particular $\int |\nabla\phi(x)|^2 \mathrm{d}x$) expresses the "effort to confine the Brownian motion".

$$P_0\left(\sup_{0\le s\le t} |B_s| < r(t)\right) \approx \exp\{-tr(t)^{-2}\}.$$

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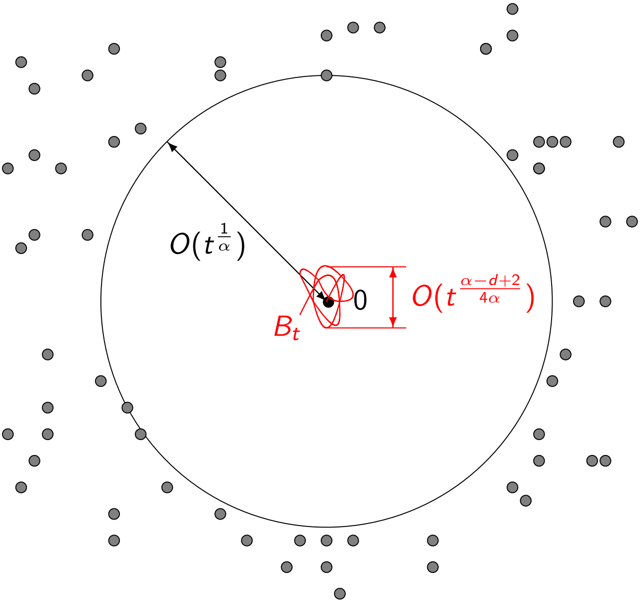
$$tr(t)^{-2} = t^{\frac{\alpha+d-2}{2\alpha}} \Leftrightarrow r(t) = t^{\frac{\alpha-d+2}{4\alpha}}.$$


$O(t^{\frac{1}{\alpha}})$
$O(t^{\frac{\alpha-d+2}{4\alpha}})$
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B_t

Main Theorem

$$Q_t\left(B_{[0,t]} \subset B\left(0, t^{\frac{\alpha-d+2}{4\alpha}}(\log t)^{\frac{1}{2}+\epsilon}\right)\right) \xrightarrow{t\to\infty} 1,$$

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$$\left\{t^{-\frac{\alpha-d+2}{4\alpha}} B_{t^{\frac{\alpha-d+2}{2\alpha}} s}\right\}_{s\geq 0} \xrightarrow{\text{in law}} \text{OU-process with}$$

"random center".

Thank you!

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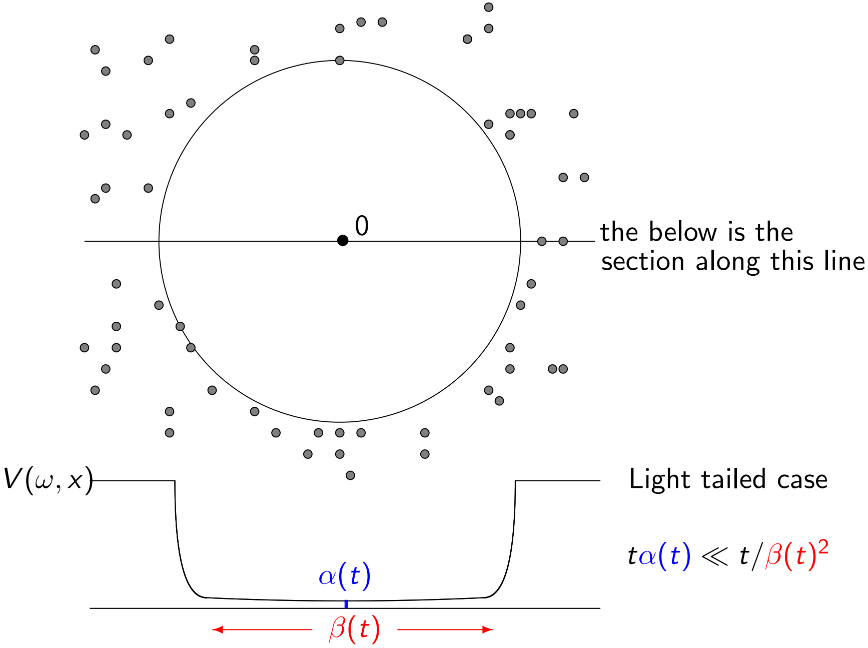
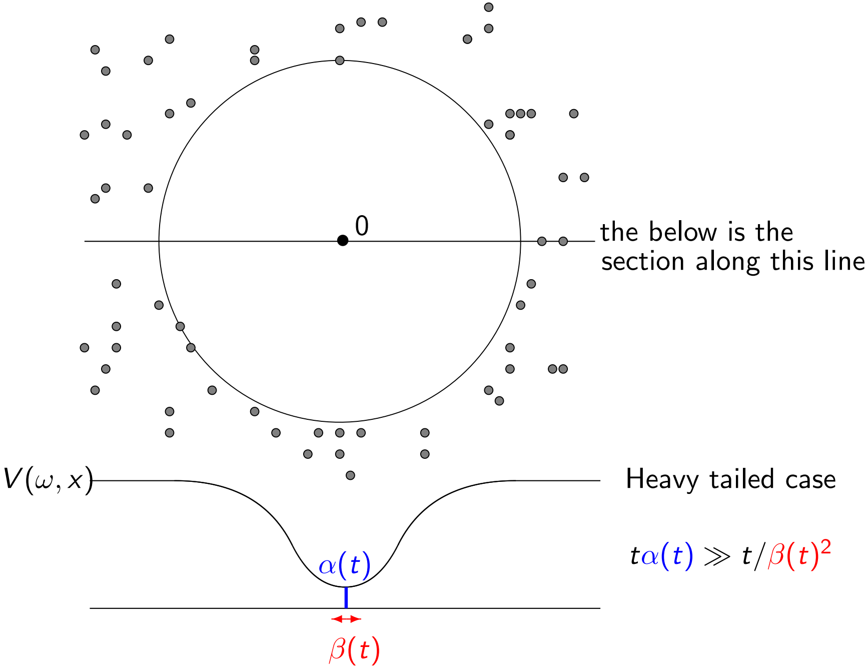
Happy birthday professor Funaki!


0
the below is the section along this line
$V(\omega, x)$
Light tailed case
$t\alpha(t) \ll t/\beta(t)^2$
$\alpha(t)$
$\beta(t)$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the below is the
section along this line
$V(\omega, x)$
Heavy tailed case
$t\alpha(t) \gg t/\beta(t)^2$
$\alpha(t)$
$\beta(t)$