Convergence of mixing times for sequences of random walks on finite graphs

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(Happy 60th birthday, Funaki-san!)

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1 Introduction

$G = (V(G), E(G))$: finite connected graph

$(X^G_m)_{m\geq 0}$: irreducible MC with trans. prob. $P_G(x, y)$, stat. prob. meas. $\pi^G(\cdot)$

$p^G_m(x, y) := \mathbf{P}^G_x(X_m = y)/\pi^G(\{y\})$: the transition density of X^G w.r.t. π^G.

For $p \in [1, \infty]$, define **the L^p-mixing time** of G by

$$t^p_{\text{mix}}(G) := \inf \left\{ m > 0 : \sup_{x\in V(G)} D^G_p(x, m) \leq 1/4 \right\},$$

where $D^G_p(x, m) := \|(p^G_m(x, \cdot) + p^G_{m+1}(x, \cdot))/2 - 1\|_{L^p(\pi^G)}$.

(Prob.) Given a sequence of graphs $(G^N)_{N\geq 1}$, obtain asymptotic behavior of $t^p_{\text{mix}}(G^N)$!

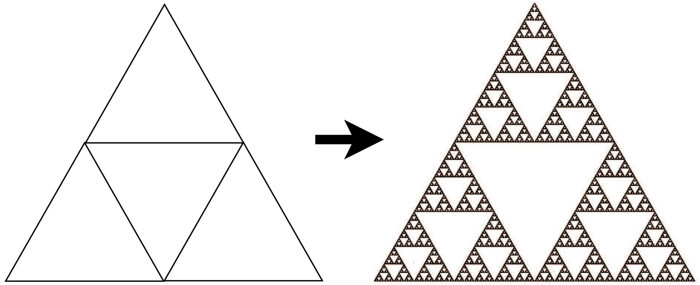
When does it converge as $N \to \infty$?

Example 0: Simple RW on $\{1,2,\cdots,N\}^d$. $(N^{-1}X^N_{[N^2t]})_{t\geq 0} \longrightarrow (B_t^{[0,1]^d})_{t\geq 0}$

$$\Rightarrow N^{-2}t^p_{\rm mix}(\{1,2,\cdots,N\}^d) \to t^p_{\rm mix}([0,1]^d).$$

Example 1: Fractal graphs (for simplicity pre-Sierpinski gasket)

G^N: pre-SG, $\{\mu^N_{xy}\}$ random (i.i.d.) conductance $\mu^N_{xy}\in[c_1,c_2]$, X^N: corresponding MC



$$(2^{-N}X^N_{[5^N t]})_{t\geq 0} \longrightarrow (B_t^F)_{t\geq 0} \quad \text{in prob.,}$$

where F is the gasket, B^F is BM on F (K-Kusuoka '96).

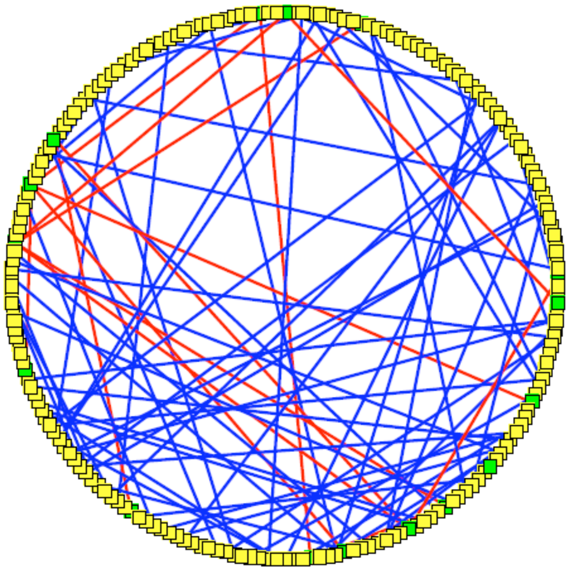
$$\Rightarrow \text{Using Theorem 2.2, } 5^{-N}t^p_{\rm mix}(G^N) \to t^p_{\rm mix}(F) \quad \text{in prob..}$$

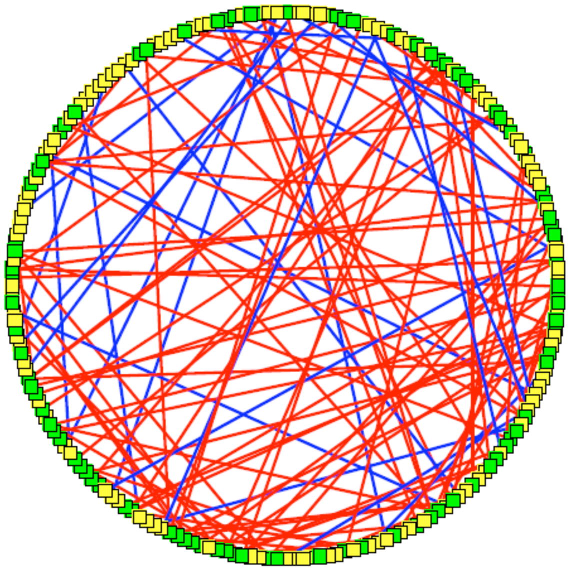
Example 3: Erdös-Rényi random graph at critical window

$G(N,p)$: Erdös-Rényi random graph   I.e. $V_N := \{1, 2, \cdots, N\}$ labeled vertices

Each $\{i, j\}$ $(i, j \in V_N)$ is connected by a bond with prob. $p \sim c/N$.

E.g. $N = 200, c = 0.8$ $N = 200, c = 1.2$ Pictures by C. Goldschmidt.





$\mathcal{C}^N$: largest connected component

$$c < 1 \Rightarrow \sharp\mathcal{C}^N = O(\log N), \quad c > 1 \Rightarrow \sharp\mathcal{C}^N \asymp N, \quad c = 1 \Rightarrow \sharp\mathcal{C}^N \asymp N^{2/3}$$

Finer scaling (critical window): $p = 1/N + \lambda N^{-4/3}$ for fixed $\lambda \in \mathbb{R}$

$\Rightarrow$ all components have size $\Theta(N^{2/3})$.

Theorem 1.1 *(Nachmias-Peres: AOP '08)* $\forall \epsilon > 0$, $\exists A = A(\epsilon, \lambda) < \infty$ *s.t.*

$$P(t^1_{\mathrm{mix}}(\mathcal{C}^N) \notin [A^{-1}N, AN]) < \epsilon \qquad \forall N >> 1.$$

Using Thm 2.2 (to be precise Thm 4.3) we can obtain the following.

Theorem 1.2 *Fix* $p \in [1, \infty]$. *If* $t^p_{\mathrm{mix}}(\rho^N)$ *is the* L^p*-mixing time of MC on* $\mathcal{C}^N$ *started from its root* ρ^N*, then*

$$N^{-1} t^p_{\mathrm{mix}}(\rho^N) \to t^p_{\mathrm{mix}}(\rho), \qquad \textit{in distribution},$$

where $t^p_{\mathrm{mix}}(\rho) \in (0, \infty)$ *is the* L^p*-mixing time of the BM on* $\mathcal{M}$ *started from* ρ*.*

Rem. We believe $N^{-1} t^p_{\mathrm{mix}}(\mathcal{C}^N) \to t^p_{\mathrm{mix}}(\mathcal{M})$ in distri. holds.

2 Theorem

Assumption 2.1 $(G^N)_{N\geq 1}$*: sequence of finite connected graphs.*

$\exists \gamma(N) > 0, (N \geq 1)$ *s.t.* $\forall I$ *compact interval,*

$$\left(\left(V(G^N), d_{G^N}\right), \pi^N, \left(q^N_{\gamma(N)t}(x,y)\right)_{x,y\in V(G^N), t\in I}\right) \to \left((F, d_F), \pi, (q_t(x,y))_{x,y\in F, t\in I}\right)$$

in a spectral Gromov-Hausdorff sense.

Theorem 2.2 *Assume Assumption 2.1. If* $\lim_{t\to\infty} \|q_t(x,\cdot) - 1\|_{L^p(\pi)} = 0$, $\forall x \in F$, *where* $p \in [1,\infty]$ *and* $q_t(\cdot,\cdot)$ *is the HK, then*

$$\lim_{N\to\infty} \gamma(N)^{-1} t^p_{\mathrm{mix}}(G^N) = t^p_{\mathrm{mix}}(F) \in (0,\infty). \tag{1}$$

3 Spectral Gromov-Hausdorff convergence

Limiting space (F, d_F): compact metric space

π: non-atomic Borel prob. meas. on F (full support)

$(q_t(x,y))_{x,y\in F,t>0}$: jointly cont. HK of a conservative irreducible Hunt proc. on F.

Assume $(q_t(x,y))_{x,y\in F,t>0}$ converges to stationarity in L^p-sense, i.e.

$$\lim_{t\to\infty} \|q_t(x,\cdot)-1\|_{L^p(\pi)} = 0, \qquad \forall x \in F. \tag{2}$$

Then the L^p-mixing time of F is finite, i.e.

$$t^p_{\mathrm{mix}}(F) := \inf\left\{ t>0 : \sup_{x\in F} \|q_t(x,\cdot)-1\|_{L^p(\pi)} \le 1/4 \right\} < \infty.$$

Gromov-Hausdorff distance

F, F': compact metric spaces

The Gromov-Hausdorff distance between F and F' is defined by

$$d_{GH}(F, F') = \inf_{Z,\phi,\phi'} d_H^Z(\phi(F), \phi'(F')),$$

where inf is taken over all metric spaces Z, isometric embeddings $\phi : F \to Z$, $\phi' : F' \to Z$.

d_H^Z is the Hausdorff distance on Z.

Recall for each K, K' compact subsets of Z,

$$d_H^Z(K, K') = \inf\{\varepsilon > 0 : K \subset K'_\varepsilon, K' \subset K_\varepsilon\},$$

where $K_\varepsilon = \{x \in Z : \rho(x, K) \leq \varepsilon\}$.

Incorporating meas. and HKs

F, F': compact metric spaces, π, π': Borel prob., q, q': HK on I (compact interval)

$$\Delta_I\left((F,\pi,q),(F',\pi',q')\right)$$
$$:= \inf_{Z,\phi,\phi',\mathcal{C}} \Bigg\{ d_H^Z(\phi(F),\phi'(F')) + d_P^Z(\pi\circ\phi^{-1},\pi'\circ\phi'^{-1})$$
$$+ \sup_{(x,x'),(y,y')\in\mathcal{C}} \left(d_Z(\phi(x),\phi'(x')) + d_Z(\phi(y),\phi'(y')) + \sup_{t\in I} |q_t(x,y) - q'_t(x',y')| \right) \Bigg\},$$

where inf is taken over all metric spaces $Z = (Z, d_Z)$, isometric embeddings $\phi : F \to Z$, $\phi' : F' \to Z$, and correspondences $\mathcal{C}$ between F and F'.

d_H^Z: Hausdorff distance in Z, d_P^Z: Prohorov distance between Borel prob's on Z.

$\mathcal{C}$: correspondence between F and F' $\overset{\text{Def}}{\Leftrightarrow}$ a subset of $F \times F'$ s.t.

$\forall x \in F$, $\exists x' \in F'$ s.t. $(x,x') \in \mathcal{C}$, and conversely $\forall x' \in F'$ $\exists x \in F$ s.t. $(x,x') \in \mathcal{C}$.

For $I \in (0,\infty)$ compact interval,

$\mathcal{M}_I$: collection of (equivalence class of) triples of the form (F,π,q).

Lemma 3.1 $(\mathcal{M}_I,\Delta_I)$ *is a separable metric space.*

$(F_n,\pi_n,q_n) \longrightarrow (F,\pi,q)$ in a spectral Gromov-Hausdorff sense

$\overset{\text{Def}}{\Leftrightarrow} \lim_{n\to\infty}\Delta_I((F_n,\pi_n,q_n),(F,\pi,q)) = 0, \quad \forall I:$ compact interval

Rem. Similar notion of spectral distances were introduced in the cpt Riemannian manifolds setting by Bérard-Besson-Gallot ('94) and by Kasue-Kumura ('94).

Under Assumption 2.1, we can isometrically embed everything into a common space!

Lemma 3.2 *Suppose Assumption 2.1 holds. Then,* $\forall I$ *compact interval,* $\exists$ *isometric embeddings of* $(V(G^N), d_{G^N})$, $N \geq 1$, *and* (F, d_F) *into a common space* (E, d_E) *s.t.*

$$\lim_{N\to\infty} d_H^E(V(G^N), F) = 0, \quad \lim_{N\to\infty} d_P^E(\pi^N, \pi) = 0,$$

and also,

$$\lim_{N\to\infty} \sup_{x,y\in F} \sup_{t\in I} \left| q^N_{\gamma(N)t}(g_N(x), g_N(y)) - q_t(x,y) \right| = 0.$$

Here we have identified the spaces $(V(G^N), d_{G^N})$, $N \geq 1$, *and* (F, d_F), *and the measures upon them with their isometric embeddings in* (E, d_E).

For each $x \in F$, $y := g_N(x) \in V(G^N)$ *if* $d_E(x, y) = \min\{d_E(x, z) : z \in V(G^N)\}$.

4 Sufficient conditions

Lemma 4.1 *Generator of the rev. proc. has a compact resolvent $\oplus$ spectral gap*

$$\Rightarrow \lim_{t\to\infty} \|q_t(x,\cdot)-1\|_{L^p(\pi)} = 0, \quad \forall x \in F, \forall p \in [1,\infty].$$

Proposition 4.2 *Suppose that $(V(G^N), d_{G^N})$, $N \geq 1$, and (F, d_F) can be isometrically embedded into $\exists (E, d_E)$ in such a way that*

$$\lim_{N\to\infty} d_H^E(V(G^N), F) = 0, \qquad \lim_{N\to\infty} d_P^E(\pi^N, \pi) = 0. \tag{3}$$

Assume further $\exists F^* \overset{dense}{\subset} F$ s.t. $\forall I \subset\subset (0,\infty)$, $x \in F^*$, $y \in F$, $r > 0$,

$$\lim_{N\to\infty} \mathbf{P}^{G^N}_{g_N(x)} \left(X^{G^N}_{\lfloor \gamma(N)t \rfloor} \in B_E(y,r) \right) = \int_{B_E(y,r)} q_t(x,y)\pi(dy) \text{ uniformly for } t \in I, \tag{4}$$

$$\lim_{\delta\to 0} \limsup_{N\to\infty} \sup_{\substack{x,y,z\in V(G^N):\\ d_{G^N}(y,z)\leq\delta}} \sup_{t\in I} \left| q^N_{\gamma(N)t}(x,y) - q^N_{\gamma(N)t}(x,z) \right| = 0. \tag{5}$$

$\Rightarrow$ *Assumption 2.1 holds.*

(5) looks very strict. However, when the MC is reversible and "strongly recurrent", one can verify this rather generally: $|f(x) - f(y)|^2 \leq R_{\text{eff}}(x,y)\mathcal{E}(f,f)$.

Distinguished starting point

Theorem 4.3 *Assume* $\exists \gamma(N) > 0, (N \geq 1)$ *s.t.* $\forall I$ *compact interval,*

$$((V(G^N), d_{G^N}, \rho^N), \pi^N, (q^N_{\gamma(N)t}(\rho^N, x))_{x \in V(G^N), t \in I}) \to ((F, d_F, \rho), \pi, (q_t(\rho, x))_{x \in F, t \in I})$$

in a spectral pointed Gromov-Hausdorff sense, where $\rho^N \in G^N$, $\rho \in F$.

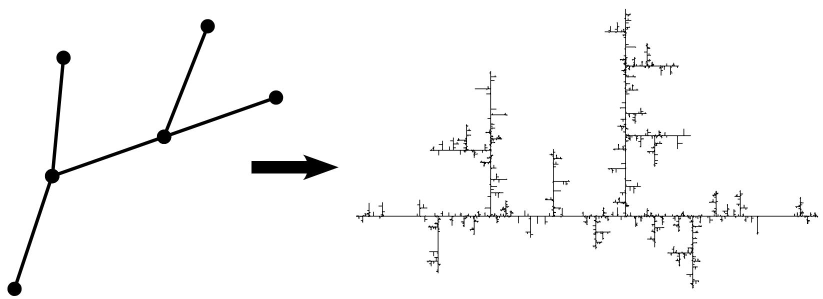
If $\lim_{t\to\infty} \|q_t(\rho, \cdot) - 1\|_{L^p(\pi)} = 0$, *where* $p \in [1, \infty]$ *and* $q_t(\cdot,\cdot)$ *is the HK, then*

$$\gamma(N)^{-1} t^{N,p}_{\text{mix}}(\rho^N) \to t^p_{\text{mix}}(\rho).$$

5 Examples

Example 2: Random trees

T^N: Galton-Watson tree with critical (mean 1) finite var offspring distri., conditioned to have N vertices, started from root ρ^N. X^N: SRW on T^N



$$(N^{-1/2}X^N_{[N^{3/2}t]})_{t\geq 0} \xrightarrow{d} (B^{\mathcal{T}}_t)_{t\geq 0},$$

where $\mathcal{T}$ is the cont. random tree (Aldous), $B^{\mathcal{T}}$ is BM on $\mathcal{T}$ (Croydon '10)

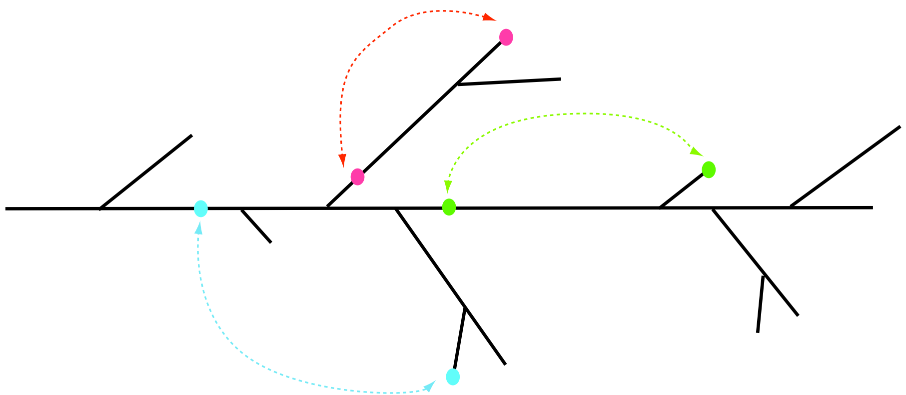
$$\Rightarrow N^{-3/2}t^p_{\rm mix}(\rho^N) \xrightarrow{d} t^p_{\rm mix}(\rho).$$

Similar results hold in infinite variance cases.

Example 3: Erdös-Rényi random graph at critical window

$N^{-1/3}\mathcal{C}^N \xrightarrow{d} \exists\mathcal{M}$ in the G-H sense (Addario-Berry, Broutin, Goldschmidt '09)

Here $\mathcal{M}$ can be constructed from a (random) real tree by gluing a (random) finite number of points as in the following figure.



X^N: SRW on $\mathcal{C}^N$ started from its root ρ^N.

$$(N^{-1/3}X^N_{[Nt]})_{t\geq 0} \xrightarrow{d} (B^{\mathcal{M}}_t)_{t\geq 0},$$

where $B^{\mathcal{M}}$ is the BM on $\mathcal{M}$ started from ρ (Croydon '10).

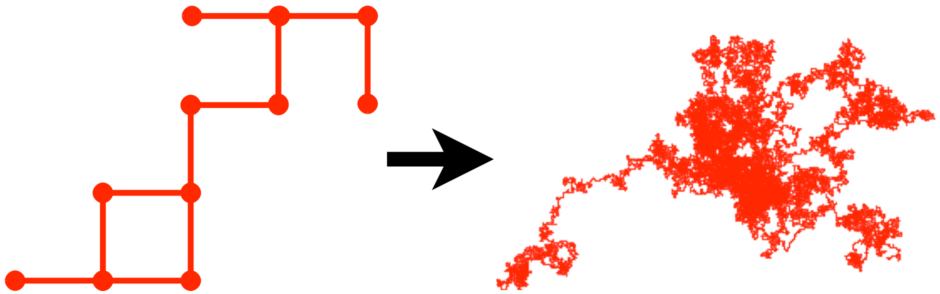
We can verify the assumption in Theorem 4.3 and prove Theorem 1.2.

Example 4: High dimensional RW trace

$\{S_n\}_n$: SRW on $\mathbb{Z}^d$ ($d \geq 5$), $G^N = S_{[0,N]}$, X^N: SRW on G^N.

$$(N^{-1} X^N_{[N^2 t]})_{t\geq 0} \xrightarrow{d} (X^{\mathcal{R}}_{ct})_{t\geq 0},$$

where $X^{\mathcal{R}}$ is the BM on $\mathcal{R} := \{B^d_t : t \in [0,1]\}$ (Croydon '09).



$$\Rightarrow cN^{-2} t^p_{\text{mix}}(G^N) \to t^p_{\text{mix}}(\mathcal{R}) \quad \text{a.s.}$$

$$\gamma(N) \text{ “}=\text{” } (\text{diam } G^N)^{d+\alpha}$$

where d is the volume growth exp. and α is the resistance growth exp.

6 Tail estimates (Reversible case)

(Q) How to obtain $\mathbf{P}\left(\gamma(N)^{-1}t^{p}_{\rm mix}(G^N) \ge \lambda\right)$, $\mathbf{P}\left(\gamma(N)^{-1}t^{p}_{\rm mix}(G^N) \le \lambda^{-1}\right)$?

General criteria (We don't need spectral G-H conv. here!) Let $R = \mathrm{diam}_d(G^N)$.

Proposition 6.1 *(1) Suppose that the following hold.*

$$\mathbf{P}(\mathrm{diam}_R(G^N) \ge \lambda R^{\alpha}) \le p_1(\lambda), \quad \mathbf{P}(\mathrm{Vol}\,(G^N) \ge \lambda R^{d}) \le p_2(\lambda).$$

Then $\mathbf{P}(t^{\infty}_{\rm mix}(G^N) \ge \lambda\gamma(N)) \le p_1(\lambda^{1/2}/8) + p_2(\lambda^{1/2})$, *where* $\gamma(N) = R^{d+\alpha}$.

(2) Suppose that the following hold.

$$\mathbf{P}(\mathrm{Vol}\,(B_R) \asymp \lambda^{\pm p_0}R^d, R_{\rm eff}(\rho^N, B_R^c) \asymp \lambda^{\pm p_1}R^{\alpha}) \ \ge\ 1 - p_1(\lambda),$$
$$\mathbf{P}(\mathrm{Vol}\,(G^N) < \lambda^{-1}R^d) \ \le\ p_2(\lambda).$$

Then $\exists c_2, c_3, p_2 > 0$ *s.t.* $\mathbf{P}(t^{1}_{\rm mix}(G^N) \le c_2\lambda^{-p_2}\gamma(N)) \le 2p_1(\lambda) + p_2(c_3\lambda)$.