Scaling limits of trap models

10th Stochastic Analysis on Large Scale Interacting Systems

Kochi University, December 5–7, 2011
60th birthday of Tadahisa Funaki

M. Jara, C. Landim, A. Teixeira

The model

- $\{G_N : N \geq 1\}$ $\quad G_N = (V_N, E_N)$ finite graphs
- $G_N = \mathbb{T}_N^d$ $\quad \mathbb{K}_N$ $\quad \mathbb{H}_N$ Random d-regular graphs
- $\{W_k : k \geq 1\}$ $\quad \sum_k W_k < \infty$

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- $x_1^N, x_2^N, \ldots, x_{|V_N|}^N$ random permutation of V_N
- $W_{x_j^N}^N = W_j$
- X_t^N Random walk on G_N with mean W_x exp. times

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- $(\mathcal{L}_N f)(x) \; = \; \frac{1}{\deg(x)} \frac{1}{W_x^N} \sum_{y \sim x} [f(y) - f(x)]$
- $\deg(x)$ degree of x $\quad y \sim x \; (x, y) \in E_N$
- Stationary state $\nu_N(x) \; \sim \; \deg(x) W^N(x)$

- W_x^N depth of trap at x Landscape of valleys

Fontes, Isopi, Newman (01)

- $\mathbb{Z}$ W_x Stable (α) $0 < \alpha < 1$
- Aging $\epsilon X(t\epsilon^{1+(1/\alpha)}) \to Z_t$ Z_t singular diffusion

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Fontes and Mathieu (08), Fontes and Lima (10)

- $X_{t\theta_N}^N$ θ_N ergodic time scale

- $\{\mathbb{X}_n : n \geq 1\}$ embedded discrete time chain
- $\mathbb{X}_n$ random walk on G_N
- Stationary state π^N $\quad \pi^N(x) \sim \deg(x)$
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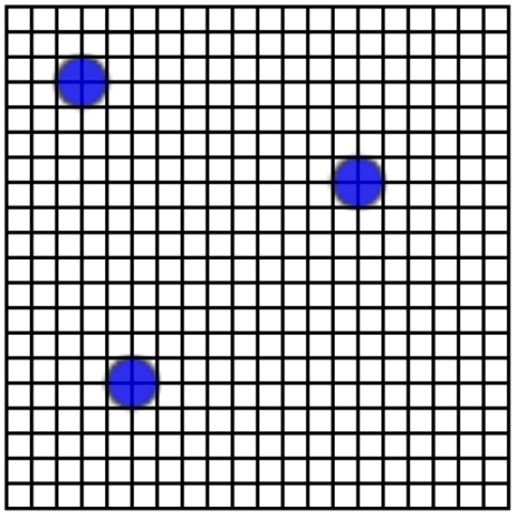
- $t_{\text{mix}} \ll$ hitting times
- $\|\mu - \nu\|_{TV} = \dfrac{1}{2} \sum_{x \in V} |\mu(x) - \nu(x)|$
- $t_{\text{mix}} = \min \left\{ n : \max_{x \in V} \|P_n(x, \cdot\,) - \pi(\,\cdot\,)\|_{TV} \leq \dfrac{1}{4} \right\}$
- $\mathbb{H}_B \ = \ \inf\{n \geq 0 : \mathbb{X}_n \in B\}$
- $\mathbb{H}_B^+ \ = \ \inf\{n \geq 1 : X_n \in B\}$
- $\mathbb{T}_N^d \qquad t_{\text{mix}} = O(N^2) \qquad \mathbb{H}_x = O(N^d) \qquad d \geq 3$

- $M_N \uparrow \infty \qquad A_N = \{x_1^N, \ldots, x_{M_n}^N\}$



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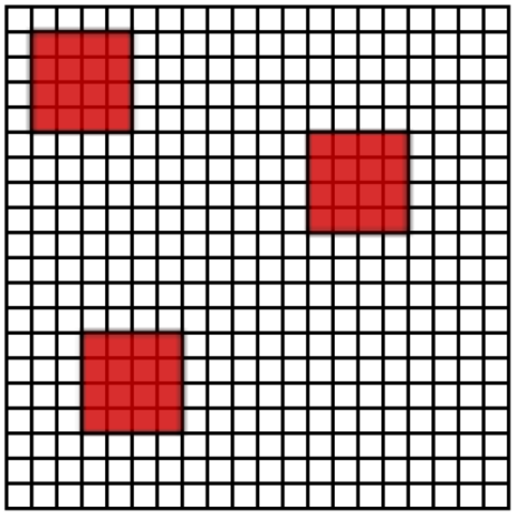

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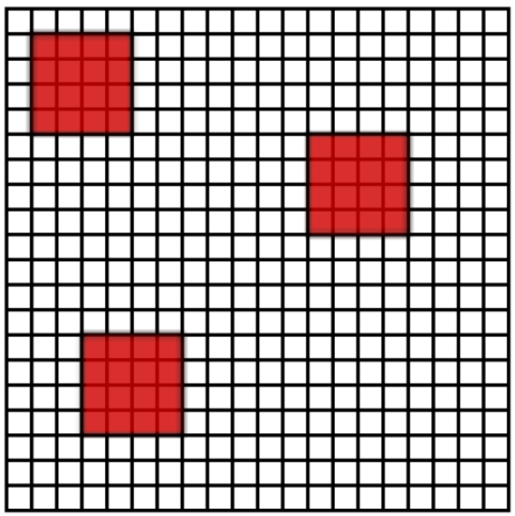
- $\ell_N \uparrow \infty \qquad B(x_j, \ell_N)$
- $B(x_j, \ell_N)$ do not overlap
- $z \notin \bigcup_j B(x_j, \ell_N) \qquad \mathbb{X}_n$ mix before hitting A_n



- $v_{\ell_N}(x) = \mathbf{P}_x\left[\mathbb{H}(B(x,\ell_n)^c) < \mathbb{H}_x^+\right]$ escape probability
- $\mathfrak{N}(x) = \#\{\mathbb{X}_n$ visits x before escaping $\}$ $\mathfrak{N}(x) \geq 1$
- $\mathfrak{N}(x) \sim$ geometric $\mathbf{P}_x[\mathfrak{N}(x) = 1] = v_{\ell_N}(x)$
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- $\Psi_N(X_{t\beta_N}^N)$ converges in some topology

K processes

- Markov process on $\mathbb{N} \cup \{\infty\}$
- $\{u_k : k \geq 1\}$ entrance measure
- $\{Z_k : k \geq 1\}$ mean exponential times
- $\sum_{k \geq 1} u_k Z_k < \infty \quad \sum_{k \geq 1} u_k = \infty$

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- Mean Z_k exponential time
- Jumps to ∞ Immediately returns to $\mathbb{N}$
- Trace of X_t on $\{1, \ldots, M\}$ $\quad p_M(i,j) = \dfrac{u_i}{\sum_{1 \leq j \leq M} u_j}$
- $\sum_{k \geq 1} u_k Z_k < \infty$

Lemma 1

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- K_t^N Markov process on $\{1,\ldots,M_N\}$
- Mean Z_k^N exponential times $\quad p_k^N$
- $(Z_k^N, B_N p_k^N) \to (Z_k, u_k)$
- $Z_k^* = \sup_N Z_k^N \quad p_k^* = \sup_N B_N p_k^N \quad \sum_{k\geq 1} p_k^*\, Z_k^* < \infty$
- Then, $K_t^N \to K_t$

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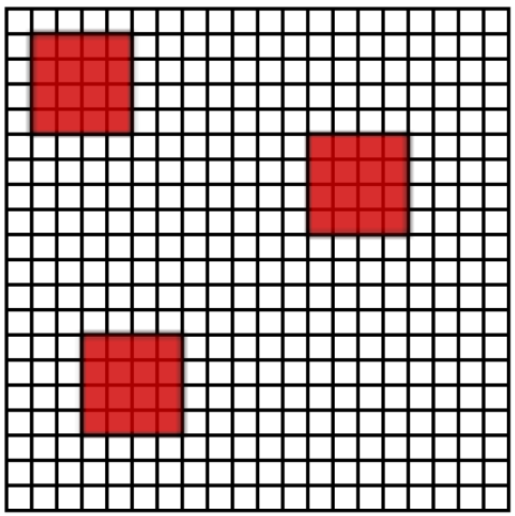
- Then, $K_t^N \to K_t$
- $W_k/v_{\ell_N}(x_k) \quad p_N(j,k)$
- $X^N_{t\beta_N} \quad \beta_N^{-1} W_k/v_{\ell_N}(x_k) \quad p_N(j,k) \sim \mathbf{P}_\pi\left[\mathbb{H}(A_N) = \mathbb{H}_{x_k}\right] =: q_N(x_k)$
- $\beta_N^{-1} W_k/v_{\ell_N}(x_k) \to Z_k \quad M_N q^N(k) \to u_k$

Lemma 2

- $q_N(x_j) \;:=\; \mathbf{P}_\pi\left[\mathbb{H}(A_N) = \mathbb{H}_{x_j}\right]$
- $A = \{x_1, \ldots, x_M\}$ For all $z \notin A$ and $L \geq 1$,

$$\sum_{j=1}^{M} \left|\mathbf{P}_z[\mathbb{X}_{\mathbb{H}_A} = x_j] - q_N(x_j)\right| \;\leq\; 2\left(2^{-L} \;+\; \mathbf{P}_z[\mathbb{H}_A < Lt_{\text{mix}}]\right)$$



Lemma 3

- $q_N(x_j) = \mathbf{P}_\pi[\mathbb{X}_{\mathbb{H}_A} = x_j]$
- $d(x_i, x_j) > 2\ell + 1 \qquad 1 \le i \ne j \le M_N$
- Then for all $L \ge 1$

$$\max_{1\le i\le M} \left| q_N(x_i) - \frac{\deg(x_i)\, v_\ell(x_i)}{\Gamma_\ell(A)} \right| \le 2 \max_{z\in R(A,\ell)} \{2^{-L} + \mathbf{P}_z[\mathbb{H}_A \le L t_{mix}]\} .$$

- $R(A,\ell) = V_N \setminus \bigcup_{j=1}^{M_N} B(x_j, \ell_N)$

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- K_t^N trace of $\Psi_N(X_t^N)$ on $\{1, \ldots, M_N\}$ $\quad K^N_{\beta_N t} \to K_t$
- $\beta_N^{-1} W_k / v_{\ell_N}(x_k) \to Z_k$ $\quad B_N \mathbf{P}_z[\mathbb{X}_{\mathbb{H}_A} = x_k] \to u_k$

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- $L_N \quad \epsilon_N = \max_{z\in R(A,\ell)}\{2^{-L_N} + \mathbf{P}_z[\mathbb{H}_A \leq L_N t_{mix}]\} \to 0$
- $\mathbf{P}_z[\mathbb{X}_{\mathbb{H}_A} = x_j] \approx \deg(x_j)\, v_\ell(x_j)\, \Gamma_\ell(A)^{-1}$
- $B_N \deg(x_j)\, v_\ell(x_j)\, \Gamma_\ell(A)^{-1} \to u_j$

Transitive graphs

- G_N transitive graphs
- $\beta_N^{-1} W_k / v_{\ell_N}(x_k) \to Z_k \quad \beta_N^{-1} W_k / v_{\ell_N}(x_1) \to Z_k$
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- Assume exists $M_N \uparrow \infty \qquad A_N = \{x_1^N, \ldots, x_{M_N}^N\} \qquad \ell_N \qquad L_N$
- $d(x_i^N, x_j^N) > 2\ell_N + 1$
- $\epsilon_N \;=\; \max_{z \in R(A^N, \ell_N)} \left\{ 2^{-L} + \mathbf{P}_z \left[\mathbb{H}_{A^N} < L t_{\text{mix}}^N \right] \right\} \to 0$
- Set $\beta_N = [v_{\ell_N}(x_1)]^{-1}$
- $K_{t\beta_N}^N \to K_t \qquad K$-process $\qquad Z_k = W_k \qquad u_k = 1$

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- $K^N_{t\beta_N} \to K_t \qquad K$-process $\qquad Z_k = W_k \qquad u_k = 1$
- $\Psi_N(X^N_{t\beta_N}) \to K_t \qquad K$-process $\qquad Z_k = W_k \qquad u_k = 1$

Application

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- $M_N = \log\log N \quad \ell_n = N/(\log N)^{1/4} \quad L_N = (\log N)^{1/4}$

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- $M_N = \log\log N \qquad \ell_n = N/(\log N)^{1/4} \qquad L_N = (\log N)^{1/4}$
- $\mathcal{P}\Big[\bigcup_{1\le i\ne j\le M_N} \{d(x_i,x_j) \le 2\ell_N + 1\}\Big] \to 0$

$$\le \; M_N^2 \mathcal{P}\Big[d(x_1,x_2) \le 2\ell_N + 1\Big]$$

$$\le \; \frac{M_N^2 \ell_N^2}{N^2} \longrightarrow 0$$

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$$\begin{aligned}
&\mathbf{P}_z\left[\mathbb{H}_{A^N} < L_N t_{\text{mix}}^N\right] \\
&\qquad \le \; M_N\, \mathbf{P}_0\left[\mathbb{H}_x < L_N N^2\right] \quad |x| \ge \ell_N \\
&\qquad \le \; M_N\, \mathbb{P}_0\left[\mathbb{H}_{T_N^{-1}x} < L_N N^2\right] \quad \mathbb{Z}^2 \\
&\qquad \le \; M_N\, \mathbb{P}_0\Big[\sup_{t \le L_N N^2} |X_t| \ge N L_N\Big] \;+\; M_N \sum_{|y| \le N L_N} \mathbb{P}_0\left[\mathbb{H}_y < L_N N^2\right]
\end{aligned}$$

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&M_N\, L_N^2\, \mathbb{P}_0\left[\mathbb{H}_y < L_N N^2\right] \\
&\qquad \le \; M_N\, L_N^2\, \mathbb{P}_0\left[\mathbb{H}_y < \mathbb{H}_{B(NL_N)}\right] \;+\; M_N\, L_N^2\, \mathbb{P}_0\left[\mathbb{H}_{B(NL_N)} < L_N N^2\right]
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Application

- $\mathbb{T}_N^2 \qquad t_{\mathsf{mix}}^N = O(N^2)$
- $M_N = \log\log N \qquad \ell_n = N/(\log N)^{1/4} \qquad L_N = (\log N)^{1/4}$
- $\beta_N = v_\ell(x)^{-1}$
- $v_\ell(x) = \mathbf{P}_x\left[\mathbb{H}(B(x,\ell_n)^c) < \mathbb{H}_x^+\right]$
- $(\log N)\, v_\ell(x) \to \dfrac{\pi}{2}$
- $\beta_N = \dfrac{2}{\pi} \log N$

Random d- regular graph

Černý, Teixeira, Windisch 2010

- $d(x_i^N, x_j^N) > 2\ell_N + 1$
- $\epsilon_N = \max_{z \in R(A,\ell)} \{2^{-L_N} + \mathbf{P}_z[\mathbb{H}_A \leq L_N t_{mix}]\} \to 0$
- $\beta_N^{-1} W_k / v_{\ell_N}(x_k) \to Z_k$
- $B_N \deg(x_j)\, v_\ell(x_j)\, \Gamma_\ell(A)^{-1} \to u_j$

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- $M_N = \log N \qquad \ell_N = \alpha \log N$
- $\mathcal{P}\big[B(x_j, \ell_N)$ are disjoint trees $\big] \geq 1 - \dfrac{1}{N^a}$

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- $\beta_N = 1 \qquad \beta_N^{-1} W_k / v_{\ell_N}(x_k) \to Z_k = \frac{d-1}{d-2} W_k$
- $B_N = M_N \qquad B_N \deg(x_j)\, v_\ell(x_j)\, \Gamma_\ell(A)^{-1} = B_N M_N^{-1} \to u_j = 1$
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- $\mathcal{P}\big[B(x_j, \ell_N)$ are disjoint trees $\big] \geq 1 - \dfrac{1}{N^a}$
- $d(x_i^N, x_j^N) > 2\ell_N + 1$
- $v_{\ell_N}(x_k) = \dfrac{d-2}{d-1}$

Random d- regular graph

Černý, Teixeira, Windisch 2010

- $\epsilon_N = \max_{z \in R(A,\ell)} \{2^{-L_N} + \mathbf{P}_z[\mathbb{H}_A \leq L_N t_{mix}]\} \to 0$
- $M_N = \log N \qquad \ell_N = \alpha \log N$
- $\mathcal{P}\big[B(x_j, \ell_N)$ are disjoint trees $\big] \geq 1 - \dfrac{1}{N^a}$
- $t^N_{\text{mix}} \leq C \log N$
- $\max_{z \in R(A,\ell)} \mathbf{P}_z[\mathbb{H}_A \leq C(\log N)^2] \to 0$