# On time regularity of generalized Ornstein-Uhlenbeck processes with cylindrical stable noise

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# Outline

- Problems
- Main Results
- Proofs
- Some Discussions

# 1 Problems

$$dX(t) = AX(t)dt + dL(t), \; t \geq 0. \tag{1}$$

$H$, a separable Hilbert space , $\langle \cdot, \cdot \rangle_H$.

$A$, generator of a $C_0$-semigroup on $H$, $A^*$ the adjoint operator of $A$.

$L$, Lévy process, $L = \sum_{n=1}^{\infty} \beta_n L^n(t) e_n$,

$L^n$ i.i.d., càdlàg real-valued Lévy processes.

$\{e_n\}_{n \in \mathbb{N}}$ fixed reference orthonormal basis in $H$.

$\beta_n$ a sequence of positive numbers.

**Problem :**

If the solution of Eq. (1) $(X(t))_{t\geq 0}$ takes value in $H$ for any $t$, is there a $H$-valued càdlàg modification of $X$? i.e. $\exists$ ? a $H$-valued càdlàg $(\tilde{X}_t)_{t\geq 0}$ such that,

$$\mathbb{P}(X_t = \tilde{X}_t) = 1, \quad \text{for any } \; t. \tag{2}$$

Assume that $\{e_n\}_{n\in\mathbb{N}} \subset \mathscr{D}(A^*)$, the weak solution of Eq. (1),

$$dX(t) = AX(t)dt + dL(t),\ t \geq 0.$$

can be represented by for any $n \in \mathbb{N}$,

$$d\langle X(t), e_n\rangle_H = \langle X(t), A^* e_n\rangle_H dt + \beta_n dL^n(t). \tag{3}$$

$\langle X(t), e_n\rangle_H \equiv X^n(t)$.

$L^n$, $\alpha-$stable processes, $\alpha \in (0,2)$.

## 1.1. Property of Sample Paths

**Kolmogorov's Extension Theorem:**

$\mathcal{S}$: State space.

construct distribution on $\mathcal{S}^{[0,\infty)}$.

However, this theorem does not describe the properties of sample paths.

Continuous or càdlàg modification of sample path is a fundamental property in Theory of Stochastic Processes, such as Martingale Theory, Markov Processes and Probabilistic Potential Theory and SDE.

[1] Doob, J.L. *Stochastic Processes.* John Wiley & Sons Inc., New York 1953

## 1.2. Generalized Ornstein-Uhlenbeck Processes

$$dX(t) = AX(t)dt + dL(t).$$

$L = \sum_{n=1}^{\infty} \beta_n L^n(t) e_n$, $L^n$ i.i.d., càdlàg $\alpha$-stable processes.

Modeling some heavy tail phenomenon.

**The time regularity of the process $X$ is of prime interest in the study of non-linear Stochastic PDEs.**
**And these studies of generalized O-U processes is a beginning point .**

## 1.3. $l^2$-valued O-U processes driven by Brownian motion

- $l^2$-valued O-U processes driven by Brownian motion

  [2] Iscoe, Marcus, McDonald, Talagrand, Zinn, (1990) *Ann. Proba.*

$$dx_k(t) = -\lambda_k x_k(t)dt + \sqrt{2a_k}dB_k, \quad k = 1, 2, \cdots .$$

They gave a simple but quite sharp criterion for continuity of $X_t$ in $l^2$.

**Theorem 1 in [2]** *$f(x)$ positive function on $[0,\infty)$ such that $\frac{f(x)}{x}$ nondecreasing for $x \geq x_1 > 0$ and*

$$\int_{x_1}^{\infty} \frac{dx}{f(x)} < \infty, \ \sum_k \frac{a_k}{\lambda_k} < \infty, \ \sup_k \frac{f(a_k) \vee x_1}{\lambda_k \vee 1} < \infty. \tag{4}$$

*Then, $x_t$ is continuous in $l^2$ a.s. Moreover, this result is best possible in the sense that it is false for any function $f(x)$, which satisfies all the above hypotheses with the exception that $\int_{x_1}^{\infty} \frac{dx}{f(x)} = \infty$.*

- $H$ or $B$-valued O-U processes

  [3] Millet, Smolenski (1992) *Prob. Theory Related Fields.*

### 1.3.1. O-U Eq. with Lévy noise

[4 ] Fuhrman, Röckner (2000) *Generalized Mahler semigroups: the non Gaussian case,* Potential Anal., 12(2000), 1-47.

• *There is an enlarged space* $E$, $H \subset_{\mathrm{HS}} E$, *such that* $(X(t))_{t\geq 0}$ *has a càdlàg path in* $E$.

[5 ] Priola, E., Zabczyk, J. *On linear evolution with cylindrical Lévy noise,*in: SPDE and Applications VIII, Proceedings of the Levico 2008 Conference.

• $L(t)$ *symmetric, and* $L(t) \in U \supset H$, *they give a necessary and sufficient condition of* $X_t \in H$, *for any* $t > 0$.

[6 ] Brzeźniak, Z., Zabczyk, J. *Regularity of Ornstein-Uhlenbeck processes driven by Lévy white noise,* Potential Anal. 32(2010)153-188.

• $L(t)$, *Lévy white noise obtained by subordination of a Gaussian white noise.* $L_t = W(Z(t))$, *Spatial continuity, Time irregularity.*

[7 ] Priola, E., Zabczyk, J. *Structural properties of semilinear SPDEs driven by cylindrical stable process,* Probab. Theory Related Fields, 149(2011), 97-137 [PZ11]

- *They conjectured in Section 4 in [7], If $L^n$ are symmetric $\alpha$-stable processes, $\alpha \in (0,2)$, the $H$-càdlàg property of Eq. (1) holds* **under much weaker conditions than $\sum_{n=1}^{\infty} \beta_n^{\alpha} < \infty$.**

**Remark 1.** $\sum_{n=1}^{\infty} \beta_n^{\alpha} < \infty \Leftrightarrow L(t) = \sum_{n=0}^{\infty} \beta_n L^n(t) e_n$ has $H$-càdlàg property.

**Remark 2.** In general, $L \in H \Rightarrow X$ has $H$-càdlàg path.

[8 ] Brzeźniak, Z., Goldys, B., Imkeller, P., Peszat, S., Priola, E., Zabczyk, J. *Time irregularity of generalized Ornstein-Uhlenbeck processes,*C. R. Acad. Sci. Paris, Ser. I 348(2010), 273-276. [BGIPPZ10]

$$dX(t) = AX(t)dt + dL(t),\ t \geq 0.$$

$$d\langle X(t), e_n\rangle_H = \langle X(t), A^* e_n\rangle_H dt + \beta_n dL^n(t),\ n \in \mathbb{N}. \tag{5}$$

$\langle X(t), e_n\rangle_H \equiv X^n(t).$

- **Theorem 2.1 [8]** $X$, $H$-valued process $(e_n) \in \mathscr{D}(A^*)$, $\beta_n \nrightarrow 0$, then $X$ has no $H$-càdlàg modification with probability 1.

- Question 1,2,3,4 ... ...

[9 ] Brzeźniak, Z., Otobe, Y. and Xie B. *Regularity for SPDE driven by $\alpha$-stable cylindrical noise. 2011, preprint*

- They obtained detailed results of spatial regularity and temporal integrability.

# 2 Main Results

$L = \sum_{n=1}^{\infty} \beta_n L^n(t) e_n$, $L^n$ i.i.d. real-valued Lévy processes, Lévy characteristic measure $\nu$. $\{e_n\}_{n\in\mathbb{N}} \subset \mathscr{D}(A^*)$,

$$d\langle X(t), e_n\rangle_H = \langle X(t), A^* e_n\rangle_H dt + \beta_n dL^n(t).$$

**Theorem 1** *Assume that the process $X$ in Eq. (1) has $H$-càdlàg modification, then for any $\epsilon > 0$, $\sum_{n=1}^{\infty} \nu(|y| \geq \epsilon/\beta_n) < \infty$.*

**Remark 3.** This theorem implies Theorem 2.1 in [BGIPPZ10]

$\beta_n \nrightarrow 0 \Rightarrow$ no $H$-càdlàg modification

$L^n$, i.i.d. $\alpha$-stable process. $\nu(dy) = \begin{cases} c_1 y^{-1-\alpha} dy, & y > 0, \\ c_2 |y|^{-1-\alpha} dy, & y < 0. \end{cases}$

**Theorem 2** *Assume $(L^n, n = 1, 2, \cdots)$ are i.i.d., non-trivial $\alpha$-stable processes, $\alpha \in (0, 2)$, and $S(t) = e^{At}$ satisfying $\|S(t)\|_{L(H,H)} \leq e^{\beta t}$, $\beta \geq 0$, (generalized contraction principle ), the following three assertions are equivalent:*
*(1) the process $(X(t), t \geq 0)$ in Eq. (1) has $H$-càdlàg modification;*
*(2) $\sum_{n=1}^{\infty} |\beta_n|^{\alpha} < \infty$;*
*(3) the process $L$ is a Lévy process on $H$.*

**Remark 4.** This result denies the conjecture in [PZ11]. And more, Theorem 2 does not need the assumption of symmetry of $L_n$.

much weaker than $\sum_{n=1}^{\infty} |\beta_n|^{\alpha} < \infty$.

**Remark 5.** In [BGIPPZ10],

**Question 3:** *Is the requirement of the process* $L$ *evolves in* $H$ *also necessary for the existence of* $H$*-càdlàg modification of* $X$*?*

Theorem 2 partly answers Question 3, *i.e.* at least if $L^n$, i.i.d. $\alpha$-stable processes, $L$ evolving in $H$ is a necessary condition of $X$ having $H$-càdlàg modification.

Moreover, if $A$ is self-adjoint, eigenvectors $e_n$, eigenvalues $-\lambda_n < 0$, $n \in \mathbb{N}$,

$$dX^n(t) = -\lambda_n X^n(t)dt + \beta_n dL^n(t), \quad t \geq 0, \quad n \in \mathbb{N}. \tag{6}$$

For $\delta \in \mathbb{R}$,

$$H_\delta \equiv \mathscr{D}(A^{\delta/2}) = \Big\{ x = \sum_{n=1}^{\infty} x_n e_n : \ \sum_{n=1}^{\infty} \lambda_n^{\delta} |x_n|^2 < \infty, \ x_n \in \mathbb{R} \Big\}.$$

**Proposition 3** *Assume $L^n$ are i.i.d., non-trivial $\alpha$-stable processes, $\alpha \in (0,2)$ and $X^n$ is the solution of Eq. (6). Then the following assertions are equivalent:*
*(1) the process $(X(t), t \geq 0)$ in Eq. (1) has $H_\delta$-càdlàg modification;*
*(2) $\sum_{n=1}^{\infty} |\beta_n \lambda_n^{\delta/2}|^\alpha < \infty$;*
*(3) the process L is a Lévy process on $H_\delta$.*

Furthermore, we apply Proposition 3 to Stochastic Heat Equation (S.H.E.) on $\mathcal{O} = (0, \pi)$ with $\alpha$-stable noise

$$dX(t) = \Delta X(t)dt + dL(t), \tag{7}$$

**Proposition 4** *If $\beta_n = 1$ for any $n \in \mathbb{N}$, Eq. (7) has $H_\delta$-càdlàg modification if and only if $\delta < -1/\alpha$.*

**Remark 6.** in [BGIPPZ10]

**Question 4:** *Is the process $X$ in S.H.E. $H_\delta$-càdlàg for $\delta \in [-\frac{1}{\alpha}, 0)$?*

Proposition 4 answers Question 4.

**Proposition 5** *Assume $L^n$ are i.i.d., non-trivial symmetric $\alpha$-stable processes. If $(\beta_n, n \geq 1)$ satisfies $\sum_{n=1}^{\infty} \beta_n^{\alpha}/n^2 < \infty$ and $\sum_{n=1}^{\infty} \beta_n^{\alpha} = \infty$, then there is no $H$-càdlàg modification of $(X(t), t \geq 0)$ in Eq. (7), even if for any $t > 0$, $X(t) \in H$.*

**Remark 7.** In [BGIPPZ10],

**Question 1:** *Does $\beta_n \to 0$ imply existence of a càdlàg modification of $X$?*

If we set $\beta_n = n^{-\frac{1}{\alpha}}$, then $\sum_{n=1}^{\infty} \beta_n^{\alpha}/n^2 < \infty$, $\sum_{n=1}^{\infty} \beta_n^{\alpha} = \infty$ and $\beta_n \to 0$ in Eq. (7) (S.H.E.). By Proposition 5, we give an example showing that $\beta_n \to 0$ does not imply the existence of $H$-càdlàg modification of $X$, even if for any $t > 0$, $X(t) \in H$ and the Lévy characteristic measure of $L$ supports on $H$. This is a negative answer to Question 1 .

**Remark 8.** Question 2 in [BGIPPZ10]: Is $e_n \in \mathscr{D}(A^*)$ essential for the validity of Theorem 2.1 .

We have no idea to this question.

# 3 Proofs

$$X(t) = \sum_{n=1}^{\infty} X^n(t) e_n, \quad X^n(t) = \langle X(t), e_n \rangle_H$$

**Lemma 1** *The process* $(X(t), t \geq 0)$ *is a* $H$*-càdlàg (resp. continuous) process with probability 1 if and only if for any* $n \in \mathbb{N}$*, the process* $(X^n(t), t \geq 0)$ *is càdlàg (resp. continuous) process with probability 1 and for any* $T > 0$*,*

$$\lim_{N\to\infty} \sup_{t\in[0,T]} \sum_{i=N}^{\infty} |X^i(t)|^2 = 0, \quad \textit{with probability 1.} \tag{8}$$

Set $\triangle f(t) = f(t) - f(t-)$. Noting that if $(X(t), t \geq 0)$ is a $H$-càdlàg process, then

$$\sup_{n \geq N} \sup_{t \in [0,T]} |\triangle X^n(t)| \leq 2\Big( \sup_{t \in [0,T]} \sum_{n=N}^{\infty} |X^n(t)|^2 \Big)^{1/2}$$

**Lemma 2** *Assume the process $(X(t), t \geq 0)$ is a $H$-càdlàg process with probability 1, then for any $T > 0$,*

$$\lim_{N \to \infty} \sup_{n \geq N} \sup_{t \in [0,T]} |\triangle X^n(t)| = 0, \text{ with probability 1.}$$

**Proof of Theorem 1** $X$ $H$-càdlàg property.

$$\tau_n = \inf\{t > 0 : |\beta_n \triangle L^n(t)| \geq \epsilon\}$$

$\tau_n$ independent exponential distributions with parameter $\psi_n = \nu(|y| \geq \epsilon/\beta_n)$.
Lemma 2 implies

$$\lim_{N\to\infty} \mathbb{P}\big(\tau_n \leq T, \text{ for some } n \geq N\big) = 0.$$

$$\mathbb{P}\big(\tau_n \leq T, \text{ for some } n \geq N\big) = 1 - \prod_{n\geq N} \mathbb{P}\big(\tau_n \leq T\big) = 1 - \exp\Big(-\sum_{n=N}^{\infty} \psi_n T\Big)$$

$$\sum_{n=1}^{\infty} \nu(|y| \geq \epsilon/\beta_n) = \sum_{n=1}^{\infty} \psi_n < \infty.$$

Applying Theorem 1 to $\alpha$-stable processes,

$$\nu(dy) = \begin{cases} c_1 y^{-1-\alpha} dy, & y > 0, \\ c_2 |y|^{-1-\alpha} dy, & y < 0. \end{cases}$$

Theorem 2 holds.

Key point: scaling invariant law of $\alpha$-stable law, or power law.

**Proof of Lemma 1**:

$\Leftarrow$ If

$$\lim_{N\to\infty} \sup_{t\in[0,T]} \sum_{i=N}^{\infty} |X^i(t)|^2 = 0, \quad \text{with probability } 1, \tag{9}$$

then for any $t \in [0,\infty)$, for any $\epsilon > 0$, by Eq.(9), there exists $N_{t,\omega,\epsilon} \in \mathbb{N}$ satisfying $\sup_{s\in[0,t+1]} \sum_{i=N_{t,\omega,\epsilon}}^{\infty} |X^i(s)|^2 \le \epsilon$.

$$\limsup_{s'\downarrow t} \|X(s') - X(t)\|_H^2 \tag{10}$$

$$\le \lim_{s'\downarrow t} \sum_{i=1}^{N_{t,\omega,\epsilon}} |X^i(s') - X^i(t)|^2 + 2 \sup_{s\in[0,t+1]} \sum_{i=N_{t,\omega,\epsilon}}^{\infty} |X^i(s)|^2 \le 2\epsilon. \tag{11}$$

$\Rightarrow$ • $V$ is a separable Hilbert space,

$$K \text{ is a compact set in } V$$

$$\Leftrightarrow$$

$$K \text{ is bounded, closed and,}$$

and for any orthonormal basis $\{v_n\}_{n\in\mathbb{N}}$ in $V$, for any $\epsilon > 0$, there is a $N_\epsilon \in \mathbb{N}$

$$\sup_{x\in K} \sum_{i=N_\epsilon}^{\infty} \langle x, v_i \rangle_V^2 < \epsilon.$$

• By the Proposition 1.1 in [10], for any $x \in D([0,T], H)$,

$$\{x(t), t \in [0,T]\} \cup \{x(t-), t \in [0,T]\}$$

is a compact set in $H$.

[10] Jakubowski, A. *On the Skorohod topology,* Ann. Inst. Henri Poincaré 22(1986), 263-285.

# 4 Some Discussions

## 4.1. Conclusions

We give a necessary and sufficient condition of càdlàg modification of Ornstein-Uhlenbeck process with cylindrical stable noise in a Hilbert space. By using this condition, we deny a conjecture and answer some questions.

## 4.2. Further problems

- $$dY(t) = AY(t)dt + F(Y(t))dt + dL(t).$$

$$dX(t) = AX(t)dt + dL(t)$$

Formally, let $z(t) = Y(t) - X(t)$,

$$\frac{dz(t)}{dt} = Az(t) + F\big(z(t) + X(t)\big).$$

This is a deterministic PDE with “random coefficients”.

If $z \in C([0,T], H)$, then $Y$ and $X$ have the same $H$-càdlàg property.

$$dY(t) = AY(t)dt + F\big(Y(t), \nabla Y(t)\big)dt + dL(t).$$

$$\frac{dz(t)}{dt} = Az(t) + F\big(z(t) + X(t), \nabla(z(t) + X(t))\big).$$

Difficult problems: Spatial-Temporal regularity and integrability of $X$ are necessary.

These works is in progress... ...

- $X \in B$, Banach space, ?

- Itô-Stratonovich type SPDE and interacting diffusions driven by stable processes. Time (ir)regularity ? such as Parabolic Andersen Model on $\mathbb{Z}^d$.

$$dX_i(t) = \kappa \sum_{j \in \mathbb{Z}^d} a(i,j) X_j(t) dt + X_i(t-) dL_i(t), \quad i \in \mathbb{Z}^d.$$


[11] Furuoya, T., Shiga, T., *Sample lyapunov exponent for a class of linear Markovian systems over* $\mathbb{Z}^d$. Osaka J. Math 35 (1998) 35-72

THANKS