On time regularity of generalized Ornstein-Uhlenbeck processes with cylindrical stable noise

Yong LIU , Jianliang ZHAI

School of Mathematical Sciences, Peking University

2011 SALSIS Dec. 5th 2011, Kochi University, Japan
liuyong@math.pku.edu.cn
Outline

- Problems
- Main Results
- Proofs
- Some Discussions
1 Problems

\[ dX(t) = AX(t)dt + dL(t), \quad t \geq 0. \quad (1) \]

\( H, \) a separable Hilbert space, \( \langle \cdot, \cdot \rangle_H. \)
\( A, \) generator of a \( C_0 \)-semigroup on \( H, A^* \) the adjoint operator of \( A. \)
\( L, \) Lévy process, \( L = \sum_{n=1}^{\infty} \beta_n L^n(t)e_n, \)
\( L^n \) i.i.d., càdlàg real-valued Lévy processes.
\( \{e_n\}_{n \in \mathbb{N}} \) fixed reference orthonormal basis in \( H. \)
\( \beta_n \) a sequence of positive numbers.
**Problem:**

If the solution of Eq. (1) \((X(t))_{t \geq 0}\) takes value in \(H\) for any \(t\), is there a \(H\)-valued càdlàg modification of \(X\)? i.e. \(\exists \tilde{X}\) a \(H\)-valued càdlàg \((\tilde{X}_t)_{t \geq 0}\) such that,

\[
P(X_t = \tilde{X}_t) = 1, \text{ for any } t. \tag{2}
\]
Assume that $\{e_n\}_{n \in \mathbb{N}} \subset \mathcal{D}(A^*)$, the weak solution of Eq. (1),

$$dX(t) = AX(t)dt + dL(t), \ t \geq 0.$$ 

can be represented by for any $n \in \mathbb{N}$,

$$d\langle X(t), e_n \rangle_H = \langle X(t), A^* e_n \rangle_H dt + \beta_n dL^n(t).$$  (3)

$\langle X(t), e_n \rangle_H \equiv X^n(t)$.

$L^n$, $\alpha$–stable processes, $\alpha \in (0, 2)$. 
1.1. Property of Sample Paths

Kolmogorov’s Extension Theorem:

$S$: State space.

construct distribution on $S^{[0,\infty)}$.

However, this theorem does not describe the properties of sample paths.

Continuous or càdlàg modification of sample path is a fundamental property in Theory of Stochastic Processes, such as Martingale Theory, Markov Processes and Probabilistic Potential Theory and SDE.

1.2. Generalized Ornstein-Uhlenbeck Processes

\[ dX(t) = AX(t)dt + dL(t). \]

\[ L = \sum_{n=1}^{\infty} \beta_n L^n(t)e_n, \quad L^n \text{ i.i.d., càdlàg } \alpha\text{-stable processes.} \]

Modeling some heavy tail phenomenon.

The time regularity of the process \( X \) is of prime interest in the study of non-linear Stochastic PDEs.
And these studies of generalized O-U processes is a beginning point.
1.3. \(l^2\)-valued O-U processes driven by Brownian motion

- \(l^2\)-valued O-U processes driven by Brownian motion


\[ dx_k(t) = -\lambda_k x_k(t) dt + \sqrt{2a_k} dB_k, \quad k = 1, 2, \ldots. \]

They gave a simple but quite sharp criterion for continuity of \(X_t\) in \(l^2\).

**Theorem 1 in [2]** \(f(x)\) positive function on \([0, \infty)\) such that \(\frac{f(x)}{x}\) nondecreasing for \(x \geq x_1 > 0\) and

\[
\int_{x_1}^{\infty} \frac{dx}{f(x)} < \infty, \quad \sum_k \frac{a_k}{\lambda_k} < \infty, \quad \sup_k \frac{f(a_k) \vee x_1}{\lambda_k \vee 1} < \infty. \tag{4}
\]

Then, \(x_t\) is continuous in \(l^2\) a.s. Moreover, this result is best possible in the sense that it is false for any function \(f(x)\), which satisfies all the above hypotheses with the exception that \(\int_{x_1}^{\infty} \frac{dx}{f(x)} = \infty\).

- \(H\) or \(B\)-valued O-U processes

1.3.1. O-U Eq. with Lévy noise


- There is an enlarged space $E$, $H \subset_{HS} E$, such that $(X(t))_{t \geq 0}$ has a càdlàg path in $E$.


- $L(t)$ symmetric, and $L(t) \in U \supset H$, they give a necessary and sufficient condition of $X_t \in H$, for any $t > 0$.


- $L(t)$, Lévy white noise obtained by subordination of a Gaussian white noise. $L_t = W(Z(t))$, Spatial continuity, Time irregularity.

- They conjectured in Section 4 in [7], If $L^n$ are symmetric $\alpha$-stable processes, $\alpha \in (0, 2)$, the $H$-càdlàg property of Eq. (1) holds under much weaker conditions than $\sum_{n=1}^{\infty} \beta_{n}^{\alpha} < \infty$.

**Remark 1.** $\sum_{n=1}^{\infty} \beta_{n}^{\alpha} < \infty \iff L(t) = \sum_{n=0}^{\infty} \beta_{n} L^n(t) e_n$ has $H$-càdlàg property.

**Remark 2.** In general, $L \in H \Rightarrow X$ has $H$-càdlàg path.
[8] Brzeźniak, Z., Goldys, B., Imkeller, P., Peszat, S., Priola, E., Zabczyk, J. Time ir-
348(2010), 273-276. [BGIPPZ10]

\[ dX(t) = AX(t)dt + dL(t), \ t \geq 0. \]

\[ d\langle X(t), e_n \rangle_H = \langle X(t), A^*e_n \rangle_H dt + \beta_n dL^n(t), \ n \in \mathbb{N}. \] (5)

\[ \langle X(t), e_n \rangle_H \equiv X^n(t). \]

- **Theorem 2.1** [8] \( X, H \)-valued process \((e_n) \in \mathcal{D}(A^*), \ \beta_n \to 0 \), then \( X \)
has no \( H \)-càdlàg modification with probability 1.

- Question 1,2,3,4 ... ...

- They obtained detailed results of spatial regularity and temporal integrability.
2 Main Results

\[ L = \sum_{n=1}^{\infty} \beta_n L^n(t)e_n, \] 
\[ L^n \text{ i.i.d. real-valued Lévy processes, Lévy characteristic measure } \nu. \]
\[ \{e_n\}_{n \in \mathbb{N}} \subset \mathcal{D}(A^*), \]

\[ d\langle X(t), e_n \rangle_H = \langle X(t), A^*e_n \rangle_H dt + \beta_n dL^n(t). \]

**Theorem 1** Assume that the process \( X \) in Eq. (1) has \( H \)-càdlàg modification, then for any \( \epsilon > 0 \),
\[ \sum_{n=1}^{\infty} \nu(|y| \geq \epsilon/\beta_n) < \infty. \]

**Remark 3.** This theorem implies Theorem 2.1 in [BGIPPZ10]

\( \beta_n \not\to 0 \Rightarrow \text{no } H \text{-càdlàg modification} \)
$L^n$, i.i.d. $\alpha$-stable process. $\nu(dy) = \begin{cases} c_1y^{-1-\alpha}dy, & y > 0, \\ c_2|y|^{-1-\alpha}dy, & y < 0. \end{cases}$

**Theorem 2** Assume $(L^n, n = 1, 2, \cdots)$ are i.i.d., non-trivial $\alpha$-stable processes, $\alpha \in (0, 2)$, and $S(t) = e^{At}$ satisfying $\|S(t)\|_{L(H,H)} \leq e^{\beta t}$, $\beta \geq 0$, (generalized contraction principle ), the following three assertions are equivalent:

1. the process $(X(t), t \geq 0)$ in Eq. (1) has $H$-càdlàg modification;
2. $\sum_{n=1}^{\infty} |\beta_n|^{\alpha} < \infty$;
3. the process $L$ is a Lévy process on $H$.

**Remark 4.** This result denies the conjecture in [PZ11]. And more, Theorem 2 does not need the assumption of symmetry of $L_n$.

much weaker than $\sum_{n=1}^{\infty} |\beta_n|^{\alpha} < \infty$. 
Remark 5. In [BGIPPPZ10],

**Question 3:** Is the requirement of the process $L$ evolves in $H$ also necessary for the existence of $H$-càdlàg modification of $X$?

Theorem 2 partly answers Question 3, *i.e.* at least if $L^n$, i.i.d. $\alpha$-stable processes, $L$ evolving in $H$ is a necessary condition of $X$ having $H$-càdlàg modification.
Moreover, if $A$ is self-adjoint, eigenvectors $e_n$, eigenvalues $-\lambda_n < 0$, $n \in \mathbb{N}$,

$$dX^n(t) = -\lambda_n X^n(t)dt + \beta_n dL^n(t), \quad t \geq 0, \quad n \in \mathbb{N}. \quad (6)$$

For $\delta \in \mathbb{R}$,

$$H_\delta \equiv \mathcal{D}(A^{\delta/2}) = \left\{ x = \sum_{n=1}^{\infty} x_n e_n : \sum_{n=1}^{\infty} \lambda_n^\delta |x_n|^2 < \infty, \ x_n \in \mathbb{R} \right\}. $$

**Proposition 3** Assume $L^n$ are i.i.d., non-trivial $\alpha$-stable processes, $\alpha \in (0, 2)$ and $X^n$ is the solution of Eq. (6). Then the following assertions are equivalent:

1. the process $(X(t), t \geq 0)$ in Eq. (1) has $H_\delta$-càdlàg modification;
2. $\sum_{n=1}^{\infty} |\beta_n \lambda_n^{\delta/2}|^\alpha < \infty$;
3. the process $L$ is a Lévy process on $H_\delta$. 
Furthermore, we apply Proposition 3 to Stochastic Heat Equation (S.H.E.) on \( O = (0, \pi) \) with \( \alpha \)-stable noise

\[
dX(t) = \Delta X(t) dt + dL(t), \quad (7)
\]

**Proposition 4** If \( \beta_n = 1 \) for any \( n \in \mathbb{N} \), Eq. (7) has \( H_\delta \)-càdlàg modification if and only if \( \delta < -1/\alpha \).

**Remark 6.** in [BGIPPPZ10]

**Question 4:** Is the process \( X \) in S.H.E. \( H_\delta \)-càdlàg for \( \delta \in [-\frac{1}{\alpha}, 0) \)?

Proposition 4 answers Question 4.
**Proposition 5** Assume $L^n$ are i.i.d., non-trivial symmetric $\alpha$-stable processes. If $(\beta_n, n \geq 1)$ satisfies $\sum_{n=1}^{\infty} \beta_n^\alpha / n^2 < \infty$ and $\sum_{n=1}^{\infty} \beta_n^\alpha = \infty$, then there is no $H$-càdlàg modification of $(X(t), t \geq 0)$ in Eq. (7), even if for any $t > 0$, $X(t) \in H$.

**Remark 7.** In [BGIPPZ10],

**Question 1:** Does $\beta_n \to 0$ imply existence of a càdlàg modification of $X$?

If we set $\beta_n = n^{-\frac{1}{\alpha}}$, then $\sum_{n=1}^{\infty} \beta_n^\alpha / n^2 < \infty$, $\sum_{n=1}^{\infty} \beta_n^\alpha = \infty$ and $\beta_n \to 0$ in Eq. (7) (S.H.E.). By Proposition 5, we give an example showing that $\beta_n \to 0$ does not imply the existence of $H$-càdlàg modification of $X$, even if for any $t > 0$, $X(t) \in H$ and the Lévy characteristic measure of $L$ supports on $H$. This is a negative answer to Question 1.
Remark 8. Question 2 in [BGIPPZ10]: Is $e_n \in \mathcal{D}(A^*)$ essential for the validity of Theorem 2.1.

We have no idea to this question.
3 Proofs

\[ X(t) = \sum_{n=1}^{\infty} X^n(t)e_n, \quad X^n(t) = \langle X(t), e_n \rangle_H \]

**Lemma 1** The process \((X(t), t \geq 0)\) is a H-càdlàg (resp. continuous) process with probability 1 if and only if for any \(n \in \mathbb{N}\), the process \((X^n(t), t \geq 0)\) is càdlàg (resp. continuous) process with probability 1 and for any \(T > 0\),

\[
\lim_{N \to \infty} \sup_{t \in [0,T]} \sum_{i=N}^{\infty} |X^i(t)|^2 = 0, \quad \text{with probability 1.} \quad (8)
\]
Set $\triangle f(t) = f(t) - f(t-)$. Noting that if $(X(t), t \geq 0)$ is a $H$-càdlàg process, then

$$
\sup_{n \geq N} \sup_{t \in [0,T]} |\triangle X^n(t)| \leq 2 \left( \sup_{t \in [0,T]} \sum_{n = N}^{\infty} |X^n(t)|^2 \right)^{1/2}
$$

**Lemma 2** Assume the process $(X(t), t \geq 0)$ is a $H$-càdlàg process with probability 1, then for any $T > 0$,

$$
\lim_{N \to \infty} \sup_{n \geq N} \sup_{t \in [0,T]} |\triangle X^n(t)| = 0, \text{ with probability 1}.
$$
Proof of Theorem 1 \(X\) H-càdlàg property.

\[
\tau_n = \inf \{ t > 0 : |\beta_n \triangle L^n(t)| \geq \epsilon \}
\]

\(\tau_n\) independent exponential distributions with parameter \(\psi_n = \nu(|y| \geq \epsilon/\beta_n)\). Lemma 2 implies

\[
\lim_{N \to \infty} \mathbb{P}(\tau_n \leq T, \text{ for some } n \geq N) = 0.
\]

\[
\mathbb{P}(\tau_n \leq T, \text{ for some } n \geq N) = 1 - \prod_{n \geq N} \mathbb{P}(\tau_n \leq T) = 1 - \exp \left( - \sum_{n=N}^{\infty} \psi_n T \right)
\]

\[
\sum_{n=1}^{\infty} \nu(|y| \geq \epsilon/\beta_n) = \sum_{n=1}^{\infty} \psi_n < \infty.
\]
Applying Theorem 1 to $\alpha$-stable processes,

$$\nu(dy) = \begin{cases} 
  c_1y^{-1-\alpha}dy, & y > 0, \\
  c_2|y|^{-1-\alpha}dy, & y < 0.
\end{cases}$$

Theorem 2 holds.

Key point: scaling invariant law of $\alpha$-stable law, or power law.
Proof of Lemma 1:

\[ \lim_{N \to \infty} \sup_{t \in [0,T]} \sum_{i=N}^{\infty} |X^i(t)|^2 = 0, \quad \text{with probability 1}, \quad (9) \]

then for any \( t \in [0,\infty) \), for any \( \epsilon > 0 \), by Eq.(9), there exists \( N_{t,\omega,\epsilon} \in \mathbb{N} \) satisfying \( \sup_{s \in [0,t+1]} \sum_{i=N_{t,\omega,\epsilon}}^{\infty} |X^i(s)|^2 \leq \epsilon \).

\[ \limsup_{s' \downarrow t} \|X(s') - X(t)\|_H^2 \quad (10) \]

\[ \leq \lim_{s' \downarrow t} \sum_{i=1}^{N_{t,\omega,\epsilon}} |X^i(s') - X^i(t)|^2 + 2 \sup_{s \in [0,t+1]} \sum_{i=N_{t,\omega,\epsilon}}^{\infty} |X^i(s)|^2 \leq 2\epsilon. \quad (11) \]
\[ \Rightarrow \quad \bullet \text{ } V \text{ is a separable Hilbert space,} \\
\]

\[ K \text{ is a compact set in } V \]

\[ \Leftrightarrow \]

\[ K \text{ is bounded, closed and}, \]

and for any orthonormal basis \( \{v_n\}_{n \in \mathbb{N}} \) in \( V \), for any \( \epsilon > 0 \), there is a \( N_\epsilon \in \mathbb{N} \)

\[
\sup_{x \in K} \sum_{i=1}^{\infty} \langle x, v_i \rangle^2_V < \epsilon.
\]

\[ \bullet \text{ By the Proposition 1.1 in [10], for any } x \in D([0, T], H), \]

\[ \{x(t), t \in [0, T]\} \cup \{x(t-), t \in [0, T]\} \]

is a compact set in \( H \).

4 Some Discussions

4.1 Conclusions

We give a necessary and sufficient condition of càdlàg modification of Ornstein-Uhlenbeck process with cylindrical stable noise in a Hilbert space. By using this condition, we deny a conjecture and answer some questions.
4.2. Further problems

\[ dY(t) = AY(t)dt + F(Y(t))dt + dL(t). \]
\[ dX(t) = AX(t)dt + dL(t) \]

Formally, let \( z(t) = Y(t) - X(t), \)
\[ \frac{dz(t)}{dt} = Az(t) + F(z(t) + X(t)). \]

This is a deterministic PDE with “random coefficients”.

If \( z \in C([0, T], H), \) then \( Y \) and \( X \) have the same \( H \)-càdlàg property.
\[ dY(t) = AY(t)dt + F(Y(t), \nabla Y(t))dt + dL(t). \]
\[ \frac{dz(t)}{dt} = Az(t) + F(z(t) + X(t), \nabla (z(t) + X(t))). \]

Difficult problems: Spatial-Temporal regularity and integrability of \( X \) are necessary.

These works is in progress... ...
• $X \in B$, Banach space, ?

• Itô-Stratonovich type SPDE and interacting diffusions driven by stable processes. Time (ir)regularity ? such as Parabolic Andersen Model on $\mathbb{Z}^d$.

$$dX_i(t) = \kappa \sum_{j \in \mathbb{Z}^d} a(i,j)X_j(t)dt + X_i(t-)dL_i(t), \quad i \in \mathbb{Z}^d.$$ 

THANKS