Hydrodynamic limit for the Ginzburg-Landau $\nabla\phi$ interface model with a conservation law and the Dirichlet boundary condition

Takao Nishikawa

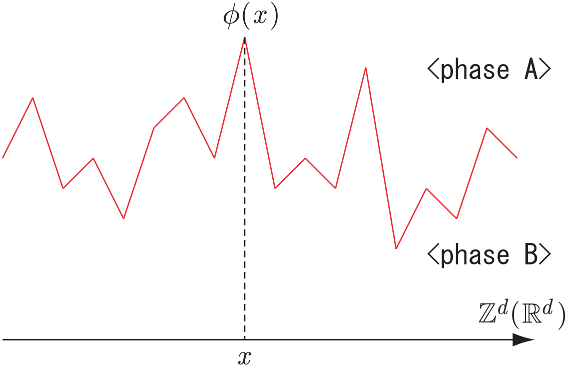
Nihon University

December 6, 2011

10th workshop on Stochastic Analysis on Large Scale Interacting Systems

Microscopic interface

Interface $\phi = \{\phi(x) \in \mathbb{R};\ x \in \mathbb{Z}^d\}$



$\phi(x)$: the height at position x

Energy of miscroscopic interface

Energy of the microscopic interface $\phi = \{\phi(x) \in \mathbb{R};\ x \in \mathbb{Z}^d\}$

$$H(\phi) = \frac{1}{2} \sum_{x,y \in \mathbb{Z}^d, |x-y|=1} V(\phi(x) - \phi(y))$$

($V : \mathbb{R} \to \mathbb{R}$ is C^2, symm., $\|V''\|_\infty < \infty$)

Dynamics - Langevin equation

Langevin eq.

$$d\phi_t(x) = -\frac{\partial H}{\partial \phi(x)}(\phi_t)dt + \sqrt{2}dw_t(x), \tag{1}$$

for

$$x \in \Gamma_N = (\mathbb{Z}/N\mathbb{Z})^d \quad \text{with periodic b.c.}$$
$$x \in D_N = ND \cap \mathbb{Z}^d \quad \text{with Dirichlet b.c.}$$

- $w = \{w_t(x); x \in \Gamma_N\}$: independent 1D B.m.'s
- $\frac{\partial H}{\partial \phi(x)} = \sum_{y:|x-y|=1} V'(\phi(x) - \phi(y))$

Dynamics - Langevin equation

Langevin eq.

$$d\phi_t(x) = -\frac{\partial H}{\partial \phi(x)}(\phi_t)dt + \sqrt{2}dw_t(x), \qquad (1)$$

for

$$x \in \Gamma_N = (\mathbb{Z}/N\mathbb{Z})^d \text{ with periodic b.c.}$$
$$x \in D_N = ND \cap \mathbb{Z}^d \text{ with Dirichlet b.c.}$$

- $w = \{w_t(x); x \in \Gamma_N\}$: independent 1D B.m.'s
- $\frac{\partial H}{\partial \phi(x)} = \sum_{y:|x-y|=1} V'(\phi(x) - \phi(y))$

Hydrodynamic scaling limit (LLN)

Macroscopic interface $h^N(t, \theta)$
$(t \in [0, t], \theta \in [0, 1)^d =: \mathbb{T}^d$ or $\theta \in D)$

$$h^N(t, x/N) = N^{-1} \phi_{N^2 t}(x), \quad x \in \Gamma_N$$

Theorem (Funaki-Spohn for Γ_N, N. for D_N with Dirichlet b.c.)

If V is strictly convex, i.e., there exist $c_-, c_+ > 0$ such that

$$c_- \le V''(\eta) \le c_+, \quad \eta \in \mathbb{R}$$

we have

$$h^N \longrightarrow h : \frac{\partial h}{\partial t} = \mathrm{div}\,\{(\nabla\sigma)(\nabla h)\} \qquad (2)$$

where $\sigma : \mathbb{R}^d \to \mathbb{R}$ is the surface tension introduced via thermodynamic limit.

Total surface tension

The equation (2) is the gradient flow with respect to the energy functional

$$\Sigma(h) = \int \sigma(\nabla h(\theta))\, d\theta \tag{3}$$

in L^2-space. The functional Σ is called ”total surface tension,” which gives the total energy of the interface h.

Remark

The assumption “V is strictly convex” can be relaxed. If we have the convexity of σ (see Cotar-Deuschel-Müller and Cotar-Deuschel) and the characterization of Gibbs measures for gradient fields, we can show the hydrodynamic limit. (joint work with J.-D. Deuschel and Y. Vignard)

Dynamics with a conservation law

Let us consider

$$d\phi_t(x) = \Delta \left\{ \frac{\partial H}{\partial \phi(\cdot)}(\phi_t) \right\}(x)dt + \sqrt{2}d\tilde{w}_t(x), \tag{4}$$

for

$$x \in \Gamma_N = (\mathbb{Z}/N\mathbb{Z})^d \text{ with periodic b.c.}$$
$$x \in D_N = ND \cap \mathbb{Z}^d \text{ with Dirichlet b.c.}$$

- $\tilde{w} = \{\tilde{w}_t(x); x \in \Gamma_N\}$: Gaussian process with cov.

$$E[\tilde{w}_s(x)\tilde{w}_t(y)] = -\Delta(x, y)s \wedge t$$

Dynamics with a conservation law

Let us consider

$$d\phi_t(x) = \Delta \left\{ \frac{\partial H}{\partial \phi(\cdot)}(\phi_t) \right\}(x)dt + \sqrt{2}d\tilde{w}_t(x), \qquad (4)$$

for

$$x \in \Gamma_N = (\mathbb{Z}/N\mathbb{Z})^d \text{ with periodic b.c.}$$
$$x \in D_N = ND \cap \mathbb{Z}^d \text{ with Dirichlet b.c.}$$

- $\tilde{w} = \{\tilde{w}_t(x); x \in \Gamma_N\}$: Gaussian process with cov.

$$E[\tilde{w}_s(x)\tilde{w}_t(y)] = -\Delta(x, y)s \wedge t$$

Dynamics with a conservation law

- Δ: (discrete) Laplacian

$$\Delta f(x) = \sum_{y \in \Gamma_N, |x-y|=1} (f(y) - f(x)), \quad x \in \Gamma_N$$

Remark

By Itô's formula, it is easy to see

$$\sum_{x \in \Gamma_N} \phi_t(x) \equiv \sum_{x \in \Gamma_N} \phi_0(x) \, (= \text{const.}), \quad t \geq 0, \tag{5}$$

that is, the total sum of the height variable ($\fallingdotseq$ number of particle) is conserved by this time evolution.

Dynamics with a conservation law

- Δ: (discrete) Laplacian

$$\Delta f(x) = \sum_{y \in \Gamma_N, |x-y|=1} (f(y) - f(x)), \quad x \in \Gamma_N$$

Remark

By Itô's formula, it is easy to see

$$\sum_{x \in \Gamma_N} \phi_t(x) \equiv \sum_{x \in \Gamma_N} \phi_0(x) \, (= \text{const.}), \quad t \geq 0, \tag{5}$$

that is, the total sum of the height variable ($\fallingdotseq$ number of particle) is conserved by this time evolution.

Hydrodynamic scaling limit on the periodic torus

Macroscopic interface $h^N(t,\theta)(t \in [0,t], \theta \in [0,1)^d =: \mathbb{T}^d)$

$$h^N(t, x/N) = N^{-1}\phi_{N^4 t}(x), \quad x \in \Gamma_N$$

Theorem (N. 2002)

If V is strictly convex, i.e., there exist $c_-, c_+ > 0$ such that

$$c_- \le V''(\eta) \le c_+, \quad \eta \in \mathbb{R}$$

we have

$$h^N \longrightarrow h : \frac{\partial h}{\partial t} = -\Delta \operatorname{div}\{(\nabla\sigma)(\nabla h)\}$$

where $\sigma : \mathbb{R}^d \to \mathbb{R}$ is the surface tension introduced via thermodynamic limit.

Problem

What happen in the case with Dirichlet b.c.?

Hydrodynamic scaling limit on finite domain

Theorem

Let D be a finite, convex domain with Lipschitz boundary. We assume that there exists $h_0 \in H^{-1}(D)$ such that

$$\lim_{N\to\infty} E\|h^N(0) - h_0\|^2_{L^2(D)} = 0$$

We then have

$$\lim_{N\to\infty} E\|h^N(t) - h(t)\|^2_{H^1(D)^*} = 0,$$

where h is the weak solution of nonlinear PDE

$$\frac{\partial h}{\partial t} = -\Delta \operatorname{div}\{(\nabla\sigma)(\nabla h)\}. \tag{6}$$

Functional spaces

Since the solution h should satisfy

$$\langle h(t), 1\rangle \equiv \langle h_0, 1\rangle,$$

$h(t)$ runs over an affine space.

We shall thus consider the tangential space:

- $H = \left\{ h \in H^1(D)^*; \ {}_{H^1(D)^*}\langle h, 1\rangle_{H^1(D)} = 0 \right\}$
- $V = \left\{ h \in H_0^1(D); \ {}_{H^1(D)^*}\langle h, 1\rangle_{H^1(D)} = 0 \right\}$
- $V \subset H \simeq H^* \subset V^*$

Functional spaces

Since the solution h should satisfy

$$\langle h(t), 1\rangle \equiv \langle h_0, 1\rangle,$$

$h(t)$ runs over an affine space.

We shall thus consider the tangential space:

- $H = \left\{ h \in H^1(D)^*;\ {}_{H^1(D)^*}\langle h, 1\rangle_{H^1(D)} = 0 \right\}$
- $V = \left\{ h \in H_0^1(D);\ {}_{H^1(D)^*}\langle h, 1\rangle_{H^1(D)} = 0 \right\}$
- $V \subset H \simeq H^* \subset V^*$

Nonlinear partial differential operator

For $f \in H_0^1(D)$, we define $A_f : V \to V^*$ by

$$A_f(h) = -\Delta[\mathrm{div}\{(\nabla\sigma)(\nabla h + \nabla f)\}$$

(single valued, monotone, coersive operator)

Formulation of PDE

We call $h = h(t, \theta)$ the solution of PDE (6) when there exists a function $f \in H_0^1(D)$ s.t.

1. $\langle h_0, 1 \rangle = \langle f, 1 \rangle$
2. $h_f \equiv h - f : [0, T] \to V^*$ is absolutely continuous in t and

$$h_f \in C([0, T]; H) \cap L^2(0, T; V) \cap W^{1,2}([0, T], V^*)$$

3. h_f satisfies

$$h_f(t) = (h_0 - f) + \int_0^t A_f(h_f(s))\, ds \quad \text{in } V^*$$

for a.e. $t \in [0, T]$.

Limit of $h(t)$ as $t \to \infty$

The solution $h(t)$ is gradient flow with respect to

$$\Sigma(h) = \int_D \sigma(\nabla h(\theta))\, d\theta$$

under the H^{-1} metric.
The limit of $h(t)$ as $t \to \infty$ is

$$\arg\inf\{\Sigma(h);\ \langle h, 1\rangle = c\},$$

which coincides "Wulff shape" obtained by [Deuschel-Giacomin-Ioffe '00].

How to show

The proof is by H^{-1}-method in

- Funaki-Spohn, Commun. Math. Phys. ('97)
- N., Probab. J. Math. Univ. Tokyo ('02)
- N., Probab. Theory Relat. Fields ('03)

What we need to do

Following the results stated before, we have the conclusion once we have

1. a priori bounds for stochastic dynamics
2. a priori bounds for discretized equation corresponding to (6)
3. uniqueness of ergodic stationary measure
   (In [N. 03] this property plays a key role.)
4. establish local equilibrium
5. derive PDE (6)

Major issues are 2 and 3.

What we need to do

Following the results stated before, we have the conclusion once we have

1. a priori bounds for stochastic dynamics
2. a priori bounds for discretized equation corresponding to (6)
3. uniqueness of ergodic stationary measure
   (In [N. 03] this property plays a key role.)
4. establish local equilibrium
5. derive PDE (6)

Major issues are 2 and 3.

What we need to do

Following the results stated before, we have the conclusion once we have

1. a priori bounds for stochastic dynamics
2. a priori bounds for discretized equation corresponding to (6)
3. uniqueness of ergodic stationary measure
   (In [N. 03] this property plays a key role.)
4. establish local equilibrium
5. derive PDE (6)

Major issues are 2 and 3.

Notations

- $(\mathbb{Z}^d)^*$: all oriented bonds in $\mathbb{Z}^d$, i.e.

$$(\mathbb{Z}^d)^* = \{(x, y) \in \mathbb{Z}^d \times \mathbb{Z}^d; |x - y| = 1\}$$

- ∇: discrete gradient

$$\nabla\phi(b) = \phi(x) - \phi(y), \quad b = (x, y)$$

- $\mathcal{X} = \left\{\nabla\phi \in \mathbb{R}^{(\mathbb{Z}^d)^*}; \phi \in \mathbb{R}^{\mathbb{Z}^d}\right\}$

Dynamics on the gradient field

For the solution ϕ_t of SDE (1), $\eta_t = \nabla \phi_t$ satisfies

$$d\eta_t(b) = -\nabla \Delta U_\cdot(\eta_t)(b)\,dt + \sqrt{2}d\nabla \tilde{w}_t(b), \tag{7}$$

where

$$U_x(\eta) := \sum_{b:x_b=x} V'(\eta(b)).$$

The generator for (7) is given by

$$\mathscr{L} = \sum_{x\in\mathbb{Z}^d} \mathscr{L}_x,$$

$$\mathscr{L}_x = -\partial_x \Delta \partial(x) + \Delta U_\cdot(x)\partial_x,$$

$$\partial_x = 2\sum_{b:x_b=x} \frac{\partial}{\partial \eta(b)}$$

Dynamics on the gradient field

For the solution ϕ_t of SDE (1), $\eta_t = \nabla \phi_t$ satisfies

$$d\eta_t(b) = -\nabla \Delta U_{\cdot}(\eta_t)(b)\, dt + \sqrt{2} d\nabla \tilde{w}_t(b), \qquad (7)$$

where

$$U_x(\eta) := \sum_{b : x_b = x} V'(\eta(b)).$$

The generator for (7) is given by

$$\mathscr{L} = \sum_{x \in \mathbb{Z}^d} \mathscr{L}_x,$$
$$\mathscr{L}_x = -\partial_x \Delta \partial(x) + \Delta U_{\cdot}(x) \partial_x,$$
$$\partial_x = 2 \sum_{b : x_b = x} \frac{\partial}{\partial \eta(b)}$$

Stationary measures and Gibbs measures

Theorem

Let a measure μ on $\mathcal{X}$ be invariant under spatial shift and tempered, that is,

$$E^{\mu}[\eta(b)^2] < \infty, \quad b \in \left(\mathbb{Z}^d\right)^* .$$

holds. If μ is a stationary measure corresponding $\mathscr{L}$, i.e.,

$$\int_{\mathcal{X}} \mathscr{L} f(\eta)\mu(d\eta) = 0$$

holds for every $f \in C^2_{\mathrm{loc}}(\mathcal{X})$, μ is then the Gibbs measure introduced by [Funaki-Spohn].

Proof of Theorem

We shall apply the same method in [Deuschel-N.-Vignard, in preparation], which is based on [Fritz, 1982]. The key ingredient is to show

$$\lim_{n\to\infty} n^{-d} I_{\Lambda_n}(\mu|_{\Lambda_n}) = 0, \tag{8}$$

where

- $I_{\Lambda_n}(\nu) = \mathscr{E}_{\Lambda_n}(\sqrt{f}, \sqrt{f}), \quad f = \dfrac{d\nu}{d\mu_{\Lambda_n}}$
- μ_{Λ_n}: finite volume Gibbs measure on $\Lambda_n := [-n, n]^d$ with free boundary condition
- $\mathscr{E}_{\Lambda_n}$: Dirichlet form for the time evolution on Λ_n with free boundary condition

Thank you for your attention!