Spectral gap for energy exchange models with rate functions approaching zero

Makiko Sasada

Department of Mathematics, Keio University

Kochi, December 5, 2011

Background

Gaspard-Gilbert (2008,2009):

- Goal: Derivation of heat equation or Fourier law of heat conduction from Newtonian dynamics
- Model: Localized hard disks (balls) in 2 and 3 dimensions
- Two step approach
  - Derive a mesoscopic master equation (Markov jump process) from the microscopic dynamics (heuristically)
  - Derive a macroscopic HDL equation from the mesoscopic dynamics

Grigo-Khanin-Szász (2011):

- Model: Introduce a class of (mesoscopic) stochastic models including GG models
- First step of the derivation of HDL eq.: Study the spectral gap

Energy exchange model

Introduced by Grigo-Khanin-Szász

- $\mathbb{R}_+^N$: state space
- $x = (x_i)_{i=1}^N$: element of $\mathbb{R}_+^N$
- x_i : energy of particle at site i

$\{X(t)\}_{t \geq 0}$: Markov process on $\mathbb{R}_+^N$ with generator $\mathcal{L}$ acting on bounded functions $A : \mathbb{R}_+^N \to \mathbb{R}$ is

$$\mathcal{L}A(x) = \sum_{i=1}^{N-1} \Lambda(x_i, x_{i+1}) \int P(x_i, x_{i+1}, d\alpha)[A(T_{i,i+1,\alpha}x) - A(x)]$$

$$(T_{i,i+1,\alpha}x)_k = \begin{cases} \alpha(x_i + x_{i+1}) & \text{if } k = i \\ (1-\alpha)(x_i + x_{i+1}) & \text{if } k = i+1 \\ x_k & \text{if } k \neq i, i+1 \end{cases}$$

- $\Lambda : \mathbb{R}_+^2 \to \mathbb{R}_+$: rate of collision, continuous
- $P(\cdot, \cdot, d\alpha)$: probability measure on $[0, 1]$

Spectral gap

- Total energy is conserved
- $\mathcal{S}_{N,e} := \{x \in \mathbb{R}_+^N \, ; \, \frac{1}{N}\sum_{i=1}^N x_i = e\}$: microcanonical surface with average energy e
- Assume that there exists a reversible measure $\pi_{N,e}$ for X_t on each $\mathcal{S}_{N,e}$

Dirichlet form associated to $\pi_{N,e}$:

$$\mathcal{D}_{N,e}(A) := \int \pi_{N,e}(dx)[-\mathcal{L}A](x)A(x)$$

$$\lambda(N,e) := \inf_A \left\{ \frac{\mathcal{D}_{N,e}(A)}{E_{\pi_{N,e}[A^2]}} \Big| E_{\pi_{N,e}}[A] = 0, \; A \in L^2(\pi_{N,e}) \right\}.$$

$\lambda(N,e)$: spectral gap of $-\mathcal{L}$ on $\mathcal{S}_{N,e}$

Spectral gap

Grigo-Khanin-Szász showed

There exists a positive constant C such that for all $N \geq 2$ and $e > 0$,

$$\lambda(N, e) \geq \frac{C}{N^2}$$

under the following conditions

- There exist a constant $\Lambda^* > 0$ such that $\Lambda(E_1, E_2) \geq \Lambda^*$
- Some assumptions for P and $\pi_{N,e}$

Product reversible measure

$$\nu_{d,e}(dx) = \frac{x^{\frac{d}{2}-1} exp(-\frac{x}{e})}{e^{\frac{d}{2}}\Gamma(\frac{d}{2})} dx : \text{ Gamma dist. with parameters } d > 0, e > 0$$

Theorem (Grigo-Khanin-Szász, 2011)

If $\Lambda(E_1, E_2)$ and $P(E_1, E_2, d\alpha)$ are of the form ("mechanical form")

- $\Lambda(E_1, E_2) = \Lambda_s(E_1 + E_2)\ \Lambda_r(\frac{E_1}{E_1+E_2})$
- $P(E_1, E_2, d\alpha) = P(\frac{E_1}{E_1+E_2}, d\alpha)$

And also, if

- $P(\beta, d\alpha)$ *has a unique invariant distribution on* $[0, 1]$
- $\Lambda(E_1, E_2) > 0$ *for all* $(E_1, E_2) \in \mathbb{R}_+^2$
- *there is a product measure* $\mu(dx) = \nu(dx_1)\nu(dx_2)\cdots\nu(dx_N)$ *which is reversible for* X_t

then ν *is a Dirac measure or* $\nu = \nu_{d,e}$ *for some* $d > 0$ *and* $e > 0$.

Mesoscopic generator in the GG model

Introduced by Gaspard-Gilbert (3-dimensional case)

- $\Lambda_s(s) = s^{1/2}$, $\Lambda_r(\beta) = \frac{\sqrt{2\pi}}{6} \frac{\frac{1}{2}+\beta\vee(1-\beta)}{\sqrt{\beta\vee(1-\beta)}}$
- $P(\beta, d\alpha) = \frac{3}{2} \frac{1\wedge\sqrt{\frac{\alpha\wedge(1-\alpha)}{\beta\wedge(1-\beta)}}}{\frac{1}{2}+\beta\vee(1-\beta)} d\alpha$
- Reversible with respect to $\nu^N_{3,e}$ for all $e > 0$ (Therefore $\pi_{N,e}$ is the conditional probability of $\nu^N_{3,1}$ (or any $\nu^N_{3,e}$) on $\mathcal{S}_{N,e}$)

$$\Rightarrow \lambda(N, e) = \sqrt{e}\lambda(N, 1) \quad \text{(by change of variables)}$$

Question $\exists C > 0 \ s.t. \ \lambda(N, e) \geq \sqrt{e}\frac{C}{N^2}$??

Remark

GG model in 2-dimension is also of the mechanical form with $\Lambda_s(s) = s^{1/2}$, and reversible with respect to $\nu^N_{2,e}$ for all $e > 0$

Stick processes that scale to the porous medium equation

Introduced by Feng-Iscoe-Seppäläinen

For each $m > 0$

- $\Lambda_s(s) = s^m$, $\Lambda_r(\beta) = \beta^m + (1-\beta)^m$, $P(\beta, d\alpha) = \frac{m|\beta-\alpha|^{m-1}}{\Lambda_r(\beta)} d\alpha$
- Reversible with respect to $\nu_{2,e}^N$ for all $e > 0$
- Gradient model
- HD equation (Porous medium equation) : $\partial_t \rho = \Delta(\rho^{m+1})$

$$\Rightarrow \lambda(N, e) = e^m \lambda(N, 1) \quad \text{(by change of variables)}$$

Question $\exists C > 0$ $s.t.$ $\lambda(N, e) \geq e^m \frac{C}{N^2}$??

Assumptions and main result

- Assume that $\pi_{N,e}$ is the conditional probability of $\nu^N_{d,1}$ (or any $\nu^N_{d,e}$) on $\mathcal{S}_{N,e}$ for some $d > 0$
- Assume that $\lambda(2,e) \geq Ke^m$ for some $K > 0$ and $m > 0$ (If $\lambda(2,1) > 0$ and $\lambda(2,e) = e^m \lambda(2,1)$, then it is satisfied) (This is also "necessary condition")

Main result

Under the assumptions, if $m \geq 1$ or $m = 0$, then

$$\exists C = C(m,d,K) > 0 \ s.t. \ \forall N \geq 2 \text{ and } \forall e > 0 \ , \ \ \lambda(N,e) \geq e^m \frac{C}{N^2}$$

For $0 < m < 1$, we can reduce the problem to that of a specific simple process of the long-range version. Moreover, we have an simple sufficient criterion for the same result.

Sketch of the proof

- Reduction of the problem to a specific simple process (Averaging model)
- Comparison theorem for the Averaging model of the local version and that of the long-range version
  - Some techniques used in the study of SG for the multi-species exclusion process
- Uniform estimate of the spectral gap for the Averaging model of the long-range version with fixed average energy
  - Case 1 $m = 0 \Rightarrow$ Caputo's method (ALEA, 2009) and some computation
  - Case 2 $m \geq 1 \Rightarrow x^m$:convex $\Rightarrow$ Comparison with Case 1
  - Case 3 $0 < m < 1 \Rightarrow$ not proved but a simple sufficient criterion is obtained (Extension of Caputo's method & Comparison with Case 2)

This strategy can be applied for a wide class of models having a product reversible measure !