Replica analysis of the 1D KPZ equation

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(Based on collaborations with T. Imamura)

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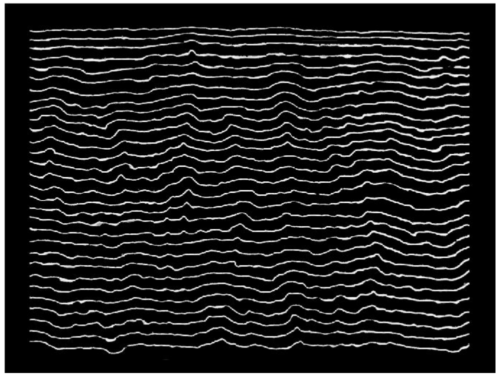
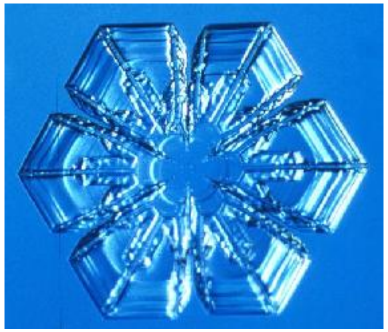
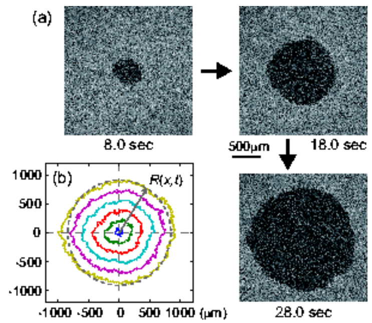
References: arxiv:1105.4659, 1111.4634

1. Introduction: 1D surface growth

- Paper combustion, bacteria colony, crystal growth, liquid crystal turbulence
- Non-equilibrium statistical mechanics
- Stochastic interacting particle systems
- Integrable systems







Kardar-Parisi-Zhang(KPZ) equation

1986 Kardar Parisi Zhang

$$\partial_t h(x,t) = \frac{1}{2}\lambda(\partial_x h(x,t))^2 + \nu\partial_x^2 h(x,t) + \sqrt{D}\eta(x,t)$$

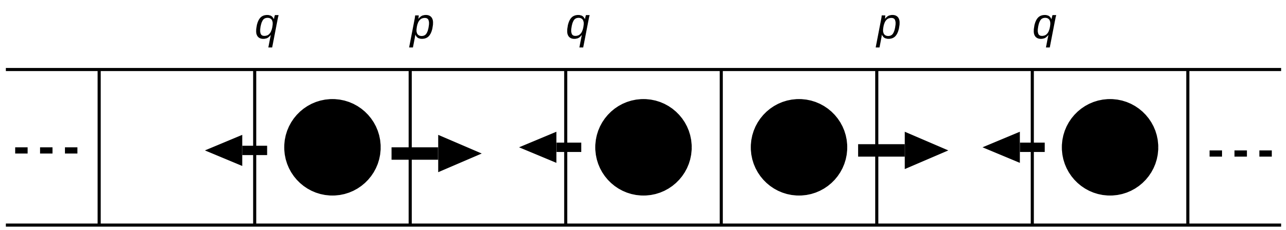
where η is the Gaussian noise with covariance
$\langle\eta(x,t)\eta(x',t')\rangle = \delta(x-x')\delta(t-t')$

- The Brownian motion is stationary.
- Dynamical RG analysis: $h(x=0,t) \simeq vt + c\xi t^{1/3}$

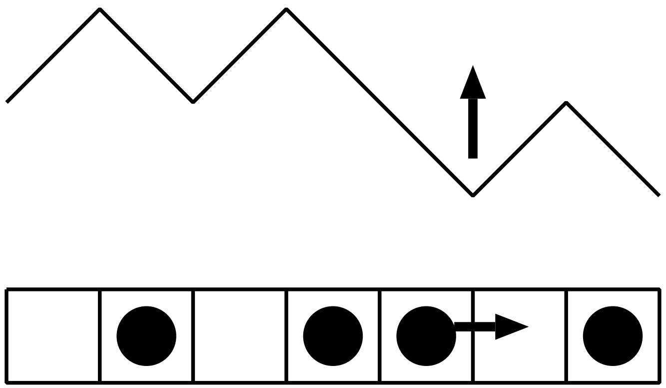
  KPZ universality class
- Now revival: New analytic and experimental developments

A discrete model: ASEP as a surface growth model

ASEP(asymmetric simple exclusion process)

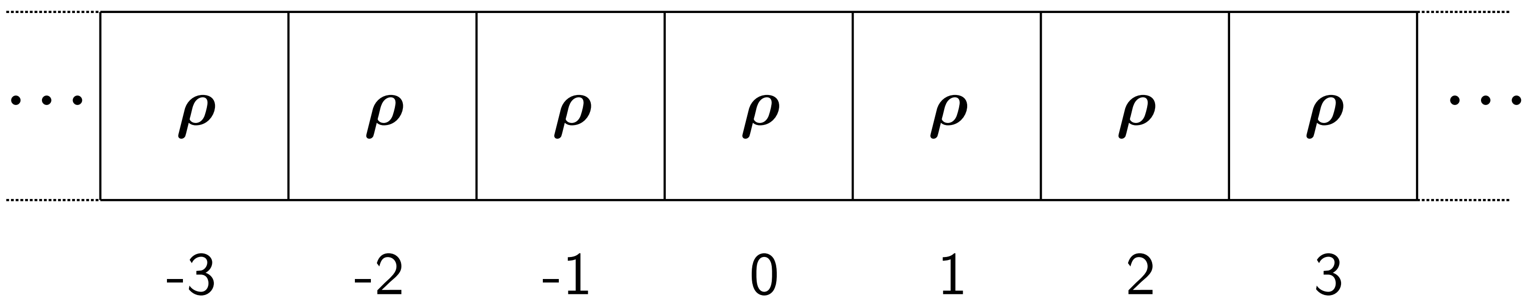


Mapping to surface growth



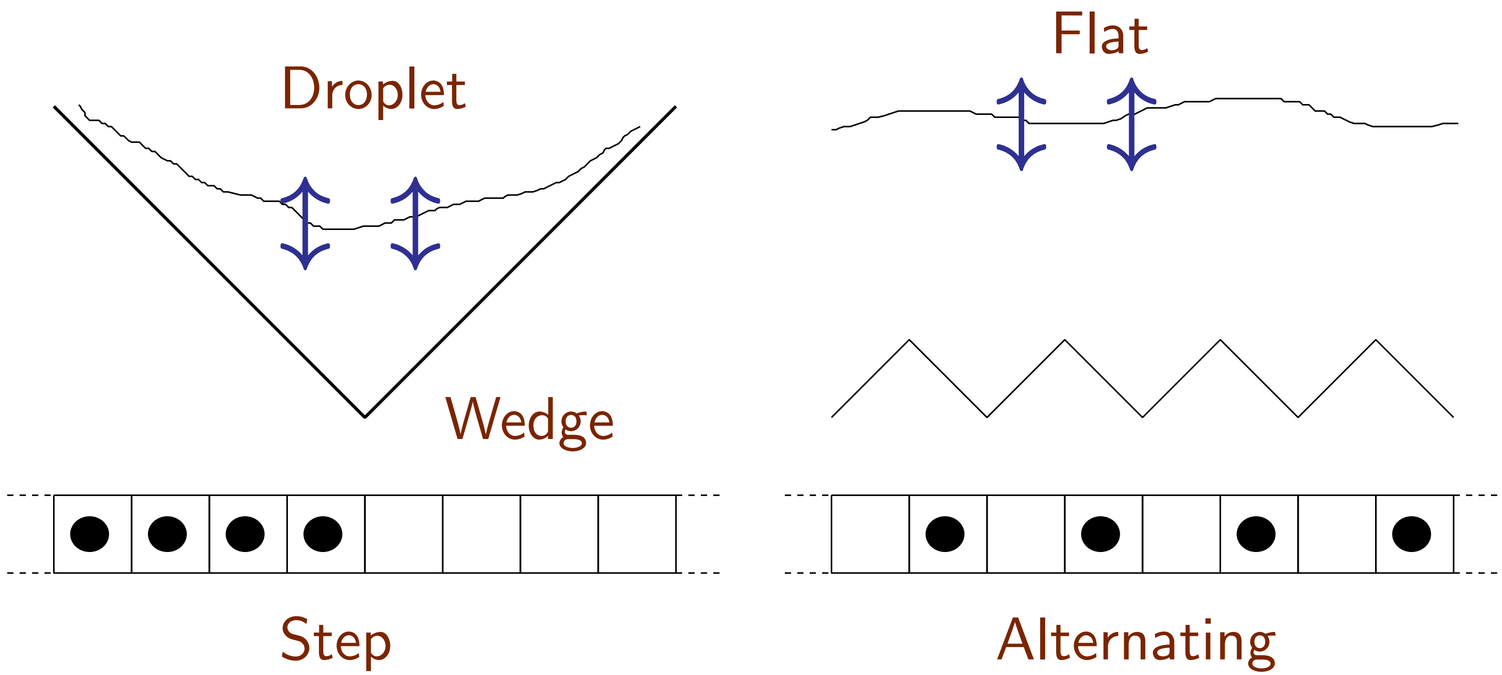
Stationary measure

ASEP $\cdots$ Bernoulli measure: each site is independent and occupied with prob. ρ $(0 < \rho < 1)$. Current is $\rho(1-\rho)$.



Surface growth $\cdots$ Random walk height profile

Surface growth and 2 initial conditions besides stationary



Integrated current $N(x,t)$ in ASEP $\Leftrightarrow$ Height $h(x,t)$ in surface growth

Current distributions for ASEP with wedge initial conditions

2000 Johansson (TASEP) 2008 Tracy-Widom (ASEP)

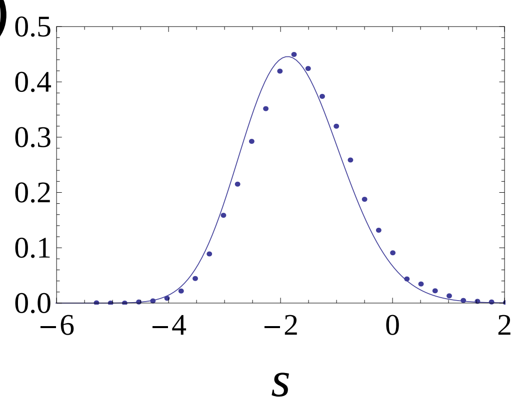
$$N(0, t/(q-p)) \simeq \frac{1}{4}t - 2^{-4/3}t^{1/3}\xi_{\mathrm{TW}}$$

Here $N(x=0,t)$ is the integrated current of ASEP at the origin and ξ_{TW} obeys the GUE Tracy-Widom distributions;

$$F_{\mathrm{TW}}(s) = \mathbb{P}[\xi_{\mathrm{TW}} \le s] = \det(1 - P_s K_{\mathrm{Ai}} P_s)$$

where K_{Ai} is the Airy kernel

$$K_{\mathrm{Ai}}(x,y) = \int_0^\infty \mathrm{d}\lambda \mathrm{Ai}(x+\lambda)\mathrm{Ai}(y+\lambda)$$



Current Fluctuations of ASEP with flat initial conditions: GOE TW distribution

More generalizations: stationary case: $\boldsymbol{F_0}$ distribution, multi-point fluctuations, etc

They can be measured experimentally!

The KPZ equation itself can be treated analytically!

Random matrix theory

GUE (Gaussian Unitary Ensemble) hermitian matrices

$$A = \begin{bmatrix} u_{11} & u_{12} + iv_{12} & \cdots & u_{1N} + iv_{1N} \\ u_{12} - iv_{12} & u_{22} & \cdots & u_{2N} + iv_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ u_{1N} - iv_{1N} & u_{2N} - iv_{2N} & \cdots & u_{NN} \end{bmatrix}$$

$u_{jj} \sim N(0, 1/2) \quad u_{jk}, v_{jk} \sim N(0, 1/4)$

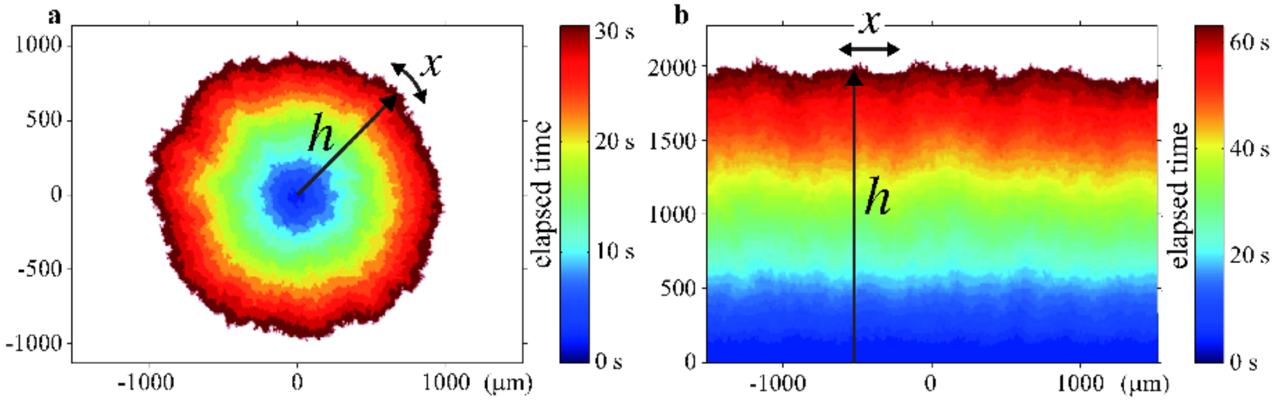
The largest eigenvalue x_{max} $\cdots$ GUE TW distribution

GOE (Gaussian Orthogonal Ensemble) real symmetric matrices

$\cdots$ GOE TW distribution

Experiments by liquid crystal turbulence

2010-2011 Takeuchi Sano



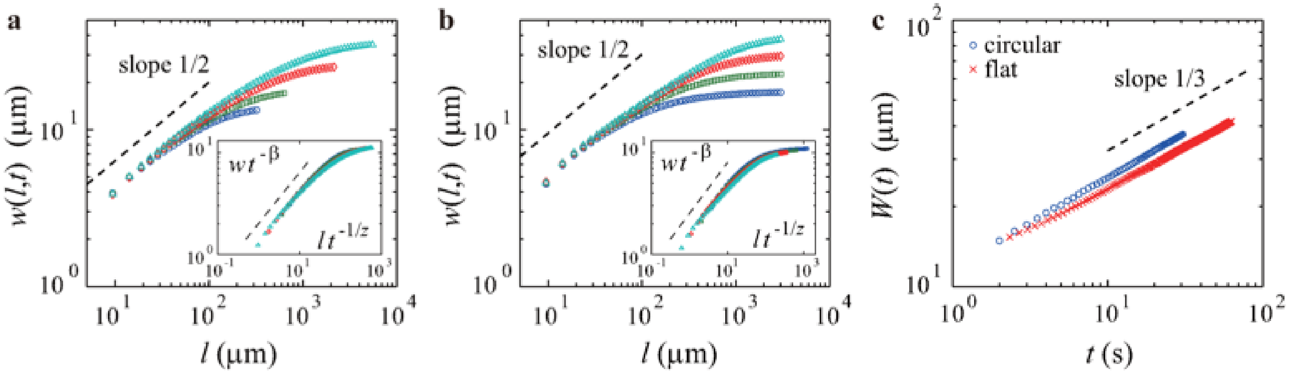


Figure 2 | Family-Vicsek scaling. a,b, Interface width $w(l, t)$ against the length scale l at different times t for the circular (a) and flat (b) interfaces. The four data correspond, from bottom to top, to $t = 2.0$ s, 4.0 s, 12.0 s and 30.0 s for the panel a and to $t = 4.0$ s, 10.0 s, 25.0 s and 60.0 s for the panel b. The insets show the same data with the rescaled axes. c, Growth of the overall width $W(t) \equiv \sqrt{\langle [h(x,t) - \langle h \rangle]^2 \rangle}$. The dashed lines are guides for the eyes showing the exponent values of the KPZ class.

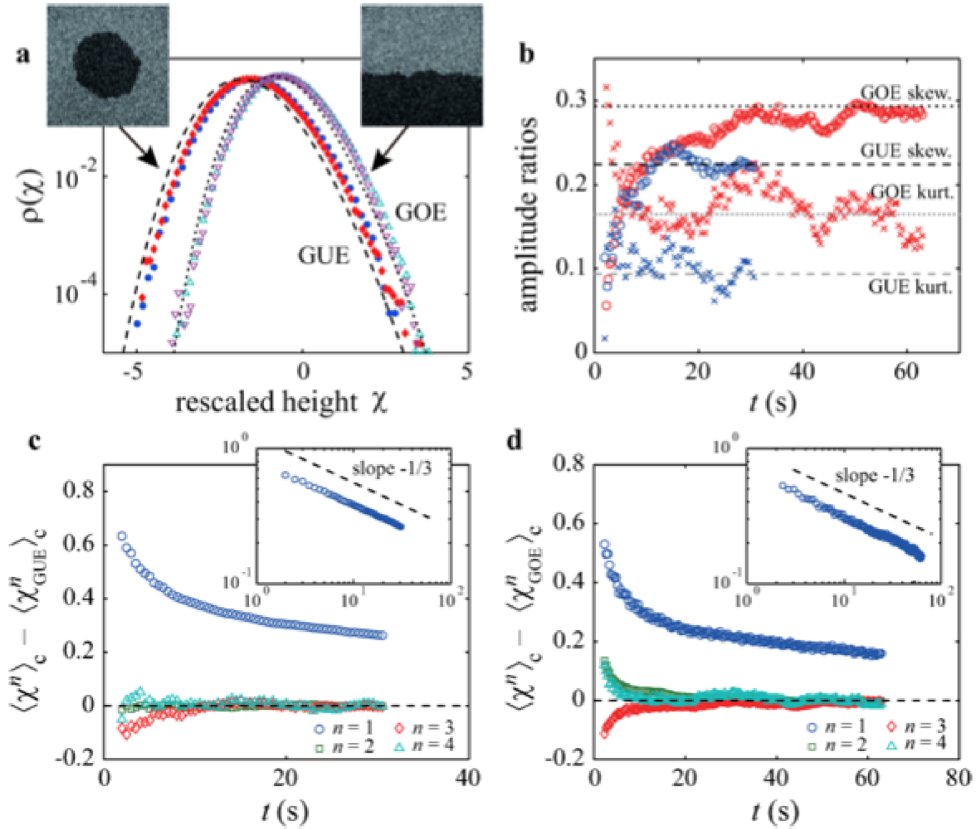


Figure 3 | Universal fluctuations. a, Histogram of the rescaled local height $\chi = (h - v_\infty t)/(\Gamma t)^{1/3}$. The blue and red solid symbols show the histograms for the circular interfaces at $t = 10$ s and 30 s; the light blue and purple open symbols are for the flat interfaces at $t = 20$ s and 60 s, respectively. The dashed and dotted curves show the GUE and GOE TW distributions, respectively. Note that for the GOE TW distribution χ is multiplied by $2^{-2/3}$ in view of the theoretical prediction[31]. b, The skewness (circle) and the kurtosis (cross) of the distribution of the interface fluctuations for the circular (blue) and flat (red) interfaces. The dashed and dotted lines indicate the values of the skewness and the kurtosis of the GUE and GOE TW distributions[31]. c, d, Differences in the cumulants between the experimental data $\langle\chi^n\rangle_c$ and the corresponding TW distributions $\langle\chi^n_{\mathrm{GUE}}\rangle_c$ for the ciruclar interfaces (c) and $\langle\chi^n_{\mathrm{GOE}}\rangle_c$ for the flat interfaces (d). The insets show the same data for $n = 1$ in logarithmic scales. The dashed lines are guides for the eyes with the slope $-1/3$.

See Takeuchi Sano Sasamoto Spohn, Sci. Rep. 1,34(2011)

The narrow wedge KPZ equation

2010 Sasamoto Spohn, Amir Corwin Quastel

- Narrow wedge initial condition
- Based on (i) the fact that the weakly ASEP is KPZ equation (1997 Bertini Giacomin) and (ii) a formula for step ASEP by 2009 Tracy Widom
- The explicit distribution function for finite t
- The KPZ equation is in the KPZ universality class

Before this

2009 Balaźs, Quastel, and Seppäläinen

The $1/3$ exponent for the stationary case

Narrow wedge initial condition

Scalings

$$x \to \alpha^2 x, \quad t \to 2\nu\alpha^4 t, \quad h \to \frac{\lambda}{2\nu} h$$

where $\alpha = (2\nu)^{-3/2}\lambda D^{1/2}$.

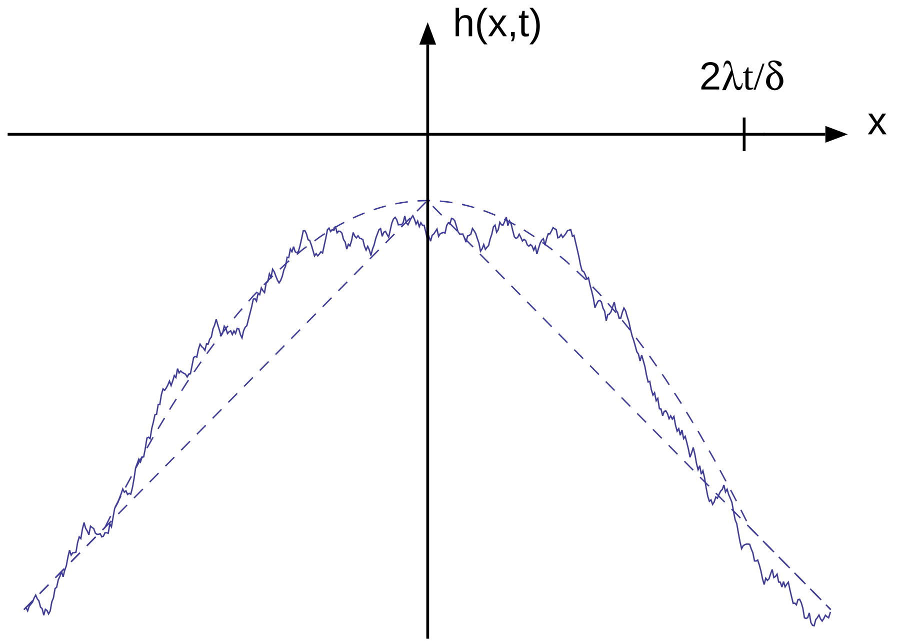
We can and will do set $\nu = \frac{1}{2}, \lambda = D = 1$.

We consider the droplet growth with macroscopic shape

$$h(x,t) = \begin{cases} -x^2/2t & \text{for } |x| \leq t/\delta\,, \\ (1/2\delta^2)t - |x|/\delta & \text{for } |x| > t/\delta \end{cases}$$

which corresponds to taking the following narrow wedge initial conditions:

$$h(x,0) = -|x|/\delta\,, \quad \delta \ll 1$$


h(x,t)
2λt/δ
x

Distribution

$$h(x,t) = -x^2/2t - \tfrac{1}{12}\gamma_t^3 + \gamma_t \xi_t$$

where $\gamma_t = (2t)^{-1/3}$.

The distribution function of ξ_t

$$F_t(s) = \mathbb{P}[\xi_t \le s] = 1 - \int_{-\infty}^{\infty} \exp\big[- \mathrm{e}^{\gamma_t(s-u)}\big]$$

$$\times\big(\det(1 - P_u(B_t - P_{\mathrm{Ai}})P_u) - \det(1 - P_u B_t P_u)\big)\mathrm{d}u$$

where $P_{\mathrm{Ai}}(x,y) = \mathrm{Ai}(x)\mathrm{Ai}(y)$.

P_u is the projection onto $[u, \infty)$ and the kernel B_t is

$$B_t(x,y) = K_{\mathrm{Ai}}(x,y) + \int_0^\infty \mathrm{d}\lambda (\mathrm{e}^{\gamma_t \lambda} - 1)^{-1}$$
$$\times \big(\mathrm{Ai}(x+\lambda)\mathrm{Ai}(y+\lambda) - \mathrm{Ai}(x-\lambda)\mathrm{Ai}(y-\lambda) \big) \,.$$

Developments (not all!)

- 2010 Calabrese Le Doussal Rosso, Dotsenko Replica
- 2010 Corwin Quastel Half-BM by step Bernoulli ASEP
- 2010 O'Connell A directed polymer model related to quantum Toda lattice
- 2010 Prolhac Spohn Multi-point distributions by replica
- 2011 Calabrese Le Dossal Flat case by replica
- 2011 Corwin et al Tropical RSK for inverse gamma polymer
- 2011 Borodin Corwin Macdonald process
- 2011 Imamura Sasamoto Half-BM and stationary case by replica

Replica analysis of KPZ equation

- Rederivation of the narrow wedge distribution by 2010 Calabrese Le Doussal Rosso, Dotsenko.

  Arrives at the correct formula by way of a divergent sum. Now there is a rigorous version for a discrete model.

- In a sense simpler than through ASEP

- Suited for generaliations

  Multipoint distributions (2010 Prolhac Spohn), Flat case (2011 Calabrese Le Dossal), Half-BM (2011 Imamura Sasamoto), Stationary case (2011 Imamura Sasamoto).

2. Stationary case

Two sided BM

$$h(x,0) = \begin{cases} B_-(-x), & x < 0, \\ B_+(x), & x > 0, \end{cases}$$

where $B_\pm(x)$ are two independent standard BMs

We consider a generalized initial condition

$$h(x,0) = \begin{cases} \tilde{B}(-x) + v_- x, & x < 0, \\ B(x) - v_+ x, & x > 0, \end{cases}$$

where $B(x), \tilde{B}(x)$ are independent standard BMs and $v_\pm$ are the strength of the drifts.

Result

For the generalized initial condition with $v_\pm$

$$F_{v_\pm,t}(s) := \mathbf{Prob}\left[h(x,t) + \gamma_t^3/12 \le \gamma_t s\right]$$
$$= \frac{\Gamma(v_+ + v_-)}{\Gamma(v_+ + v_- + \gamma_t^{-1} d/ds)}\left[1 - \int_{-\infty}^{\infty} du e^{-e^{\gamma_t(s-u)}} \nu_{v_\pm,t}(u)\right]$$

Here $\nu_{v_\pm,t}(u)$ is expressed as a difference of two Fredholm determinants,

$$\nu_{v_\pm,t}(u) = \det\left(1 - P_u(B_t^\Gamma - P_{\mathrm{Ai}}^\Gamma)P_u\right) - \det\left(1 - P_u B_t^\Gamma P_u\right),$$

where P_s represents the projection onto (s,∞),

$$P_{\mathrm{Ai}}^\Gamma(\xi_1,\xi_2) = \mathrm{Ai}_\Gamma^\Gamma\left(\xi_1, \frac{1}{\gamma_t}, v_-, v_+\right)\mathrm{Ai}_\Gamma^\Gamma\left(\xi_2, \frac{1}{\gamma_t}, v_+, v_-\right)$$

$$
\begin{aligned}
B_t^{\Gamma}(\xi_1, \xi_2) = \int_{-\infty}^{\infty} dy \frac{1}{1 - e^{-\gamma_t y}} \mathrm{Ai}_{\Gamma}^{\Gamma}\left(\xi_1 + y, \frac{1}{\gamma_t}, v_-, v_+\right) \\
\times \mathrm{Ai}_{\Gamma}^{\Gamma}\left(\xi_2 + y, \frac{1}{\gamma_t}, v_+, v_-\right),
\end{aligned}
$$

and

$$
\mathrm{Ai}_{\Gamma}^{\Gamma}(a, b, c, d) = \frac{1}{2\pi} \int_{\Gamma_{i\frac{d}{b}}} dz e^{iza + i\frac{z^3}{3}} \frac{\Gamma(ibz + d)}{\Gamma(-ibz + c)},
$$

where Γ_{z_p} represents the contour from $-\infty$ to ∞ and, along the way, passing below the pole at $z = id/b$.

Height distribution for the stationary KPZ equation

$$\boldsymbol{F_{0,t}(s) = \frac{1}{\Gamma(1+\gamma_t^{-1} d/ds)} \int_{-\infty}^{\infty} du\, \gamma_t e^{\gamma_t(s-u)+e^{-\gamma_t(s-u)}} \nu_{0,t}(u)}$$

where $\boldsymbol{\nu_{0,t}(u)}$ is obtained from $\boldsymbol{\nu_{v_\pm,t}(u)}$ by taking $\boldsymbol{v_\pm \to 0}$ limit.

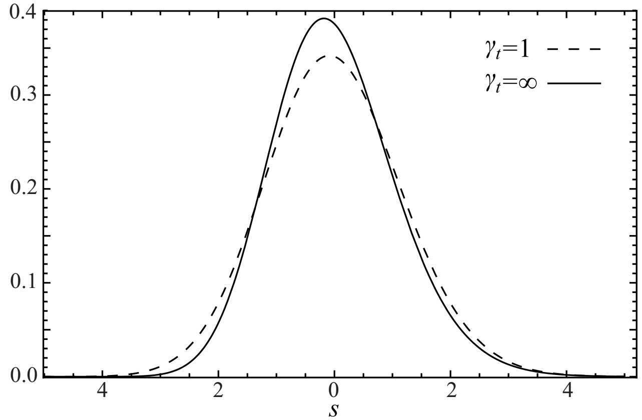


Figure 1: Stationary height distributions for the KPZ equation for $\boldsymbol{\gamma_t = 1}$ case. The solid curve is $\boldsymbol{F_0}$.

Stationary 2pt correlation function

$$C(x,t) = \langle (h(x,t) - \langle h(x,t)\rangle)^2 \rangle$$

$$g_t(y) = (2t)^{-2/3} C\left((2t)^{2/3} y, t\right)$$

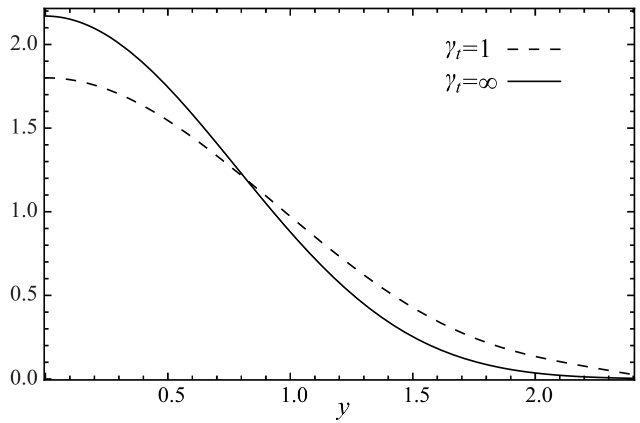


Figure 2: Stationary 2pt correlation function $g_t''(y)$ for $\gamma_t = 1$. The solid curve is the corresponding quantity in the scaling limit $g''(y)$.

Derivation

Cole-Hopf transformation

1997 Bertini and Giacomin

$$h(x,t) = \log\left(Z(x,t)\right)$$

$Z(x,t)$ is the solution of the stochastic heat equation,

$$\frac{\partial Z(x,t)}{\partial t} = \frac{1}{2}\frac{\partial^2 Z(x,t)}{\partial x^2} + \eta(x,t)Z(x,t).$$

and can be considered as directed polymer in random potential η.

cf. Hairer Well-posedness of KPZ equation without Cole-Hopf

Feynmann-Kac and Generating function

Feynmann-Kac expression for the partition function,

$$Z(x,t) = \mathbb{E}_x \left(\exp\left[\int_0^t \eta\,(b(s), t-s)\, ds \right] Z(b(t),0) \right)$$

We consider the Nth replica partition function $\langle Z^N(x,t)\rangle$ and compute their generating function $G_t(s)$ defined as

$$G_t(s) = \sum_{N=0}^{\infty} \frac{\left(-e^{-\gamma_t s}\right)^N}{N!} \left\langle Z^N(0,t) \right\rangle e^{N\frac{\gamma_t^3}{12}}$$

with $\gamma_t = (t/2)^{1/3}$.

δ-Bose gas

Taking the Gaussian average over the noise η, one finds that the replica partition function can be written as

$$\langle Z^N(x,t)\rangle$$

$$= \prod_{j=1}^{N} \int_{-\infty}^{\infty} dy_j \int_{x_j(0)=y_j}^{x_j(t)=x} D[x_j(\tau)] \exp\left[-\int_0^t d\tau \left(\sum_{j=1}^{N} \frac{1}{2} \left(\frac{dx}{d\tau} \right)^2 \right.\right.$$

$$\left.\left. - \sum_{j\neq k=1}^{N} \delta\left(x_j(\tau) - x_k(\tau)\right) \right) \right] \times \left\langle \exp\left(\sum_{k=1}^{N} h(y_k, 0) \right) \right\rangle$$

$$= \langle x | e^{-H_N t} | \Phi \rangle.$$

H_N is the Hamiltonian of the δ-Bose gas,

$$H_N = -\frac{1}{2}\sum_{j=1}^{N}\frac{\partial^2}{\partial x_j^2} - \frac{1}{2}\sum_{j\neq k}^{N}\delta(x_j - x_k),$$

$|\Phi\rangle$ represents the state corresponding to the initial condition. We compute $\langle Z^N(x,t)\rangle$ by expanding in terms of the eigenstates of H_N,

$$\langle Z(x,t)^N\rangle = \sum_z \langle x|\Psi_z\rangle\langle\Psi_z|\Phi\rangle e^{-E_z t}$$

where E_z and $|\Psi_z\rangle$ are the eigenvalue and the eigenfunction of H_N: $H_N|\Psi_z\rangle = E_z|\Psi_z\rangle$.

The state $|\boldsymbol{\Phi}\rangle$ can be calculated because the initial condition is Gaussian. For the region where $x_1 < \ldots < x_l < 0 < x_{l+1} < \ldots < x_N, 1 \le l \le N$ it is given by

$$\langle x_1, \cdots, x_N|\boldsymbol{\Phi}\rangle = e^{v_- \sum_{j=1}^{l} x_j - v_+ \sum_{j=l+1}^{N} x_j}$$

$$\times \prod_{j=1}^{l} e^{\frac{1}{2}(2l-2j+1)x_j} \prod_{j=1}^{N-l} e^{\frac{1}{2}(N-l-2j+1)x_{l+j}}$$

We symmetrize wrt $x_1, \ldots, x_N$.

Bethe states

By the Bethe ansatz, the eigenfunction is given as

$$\langle x_1, \cdots, x_N | \Psi_z \rangle = C_z \sum_{P \in S_N} \mathrm{sgn} P$$

$$\times \prod_{1 \le j < k \le N} \left(z_{P(j)} - z_{P(k)} + i\,\mathrm{sgn}(x_j - x_k) \right) \exp\left(i \sum_{l=1}^{N} z_{P(l)} x_l \right)$$

N momenta z_j $(1 \le j \le N)$ are parametrized as

$$z_j = q_\alpha - \frac{i}{2}\left(n_\alpha + 1 - 2r_\alpha\right), \quad \text{for } j = \sum_{\beta=1}^{\alpha-1} n_\beta + r_\alpha .$$

$(1 \le \alpha \le M$ and $1 \le r_\alpha \le n_\alpha)$. They are divided into M groups where $1 \le M \le N$ and the αth group consists of n_α quasimomenta $z_j' s$ which shares the common real part q_α.

$$C_z = \left(\frac{\prod_{\alpha=1}^{M} n_\alpha}{N!} \prod_{1 \le j < k \le N} \frac{1}{|z_j - z_k - i|^2} \right)^{1/2}$$

$$E_z = \frac{1}{2} \sum_{j=1}^{N} z_j^2 = \frac{1}{2} \sum_{\alpha=1}^{M} n_\alpha q_\alpha^2 - \frac{1}{24} \sum_{\alpha=1}^{M} (n_\alpha^3 - n_\alpha)\,.$$

Expanding the moment in terms of the Bethe states, we have

$$\begin{aligned}
&\langle Z^N(x,t) \rangle \\
&= \sum_{M=1}^{N} \frac{N!}{M!} \prod_{j=1}^{N} \int_{-\infty}^{\infty} dy_j \left(\int_{-\infty}^{\infty} \prod_{\alpha=1}^{M} \frac{dq_\alpha}{2\pi} \sum_{n_\alpha=1}^{\infty} \right) \delta_{\sum_{\beta=1}^{M} n_\beta, N} \\
&\quad \times e^{-E_z t} \langle x | \Psi_z \rangle \langle \Psi_z | y_1, \cdots, y_N \rangle \langle y_1, \cdots, y_N | \Phi \rangle.
\end{aligned}$$

The completeness of Bethe states was proved by Prolhac Spohn

We see

$$\langle \Psi_z | \Phi \rangle = N! C_z \sum_{P \in S_N} \mathrm{sgn} P \prod_{1 \le j < k \le N} \left(z^*_{P(j)} - z^*_{P(k)} + i \right)$$

$$\times \sum_{l=0}^{N} (-1)^l \prod_{m=1}^{l} \frac{1}{\sum_{j=1}^{m} (-i z^*_{P_j} + v_-) - m^2/2}$$

$$\times \prod_{m=1}^{N-l} \frac{1}{\sum_{j=N-m+1}^{N} (-i z^*_{P_j} - v_+) + m^2/2}.$$

Combinatorial identities

(1)

$$\sum_{P \in S_N} \mathrm{sgn} P \prod_{1 \le j < k \le N} \left(w_{P(j)} - w_{P(k)} + i f(j,k) \right)$$

$$= N! \prod_{1 \le j < k \le N} (w_j - w_k)$$

(2)For any complex numbers w_j $(1 \leq j \leq N)$ and a,

$$\sum_{P \in S_N} \mathrm{sgn} P \prod_{1 \leq j < k \leq N} \left(w_{P(j)} - w_{P(k)} + a\right)$$
$$\times \sum_{l=0}^{N} (-1)^l \prod_{m=1}^{l} \frac{1}{\sum_{j=1}^{m} (w_{P(j)} + v_-) - m^2 a/2}$$
$$\times \prod_{m=1}^{N-l} \frac{1}{\sum_{j=N-m+1}^{N} (w_{Pj} - v_+) + m^2 a/2}$$
$$= \frac{\prod_{m=1}^{N} (v_+ + v_- - am) \prod_{1 \leq j < k \leq N} (w_j - w_k)}{\prod_{m=1}^{N} (w_m + v_- - a/2)(w_m - v_+ + a/2)}.$$

A similar identity in the context of ASEP has not been found.

Generating function

$$G_t(s) = \sum_{N=0}^{\infty} \prod_{l=1}^{N} (v_+ + v_- - l) \sum_{M=1}^{N} \frac{(-e^{-\gamma_t s})^N}{M!}$$

$$\prod_{\alpha=1}^{M} \left(\int_0^{\infty} d\omega_\alpha \sum_{n_\alpha=1}^{\infty} \right) \delta_{\sum_{\beta=1}^{M} n_\beta, N}$$

$$\det \left(\int_C \frac{dq}{\pi} \frac{e^{-\gamma_t^3 n_j q^2 + \frac{\gamma_t^3}{12} n_j^3 - n_j(\omega_j + \omega_k) - 2iq(\omega_j - \omega_k)}}{\prod_{r=1}^{n_j} (-iq + v_- + \frac{1}{2}(n_j - 2r))(iq + v_+ + \frac{1}{2}(n_j - 2r))} \right)$$

where the contour is $C = \mathbb{R} - ic$ with c taken large enough.

This generating function itself is not a Fredholm determinant due to the novel factor $\prod_{l=1}^{N}(v_+ + v_- - l)$.

We consider a further generalized initial condition in which the initial overall height χ obeys a certain probability distribution.

$$\tilde{h} = h + \chi$$

where h is the original height for which $h(0,0) = 0$. The random variable χ is taken to be independent of h.

Moments
$$\langle e^{N\tilde{h}} \rangle = \langle e^{Nh} \rangle \langle e^{N\chi} \rangle.$$

We postulate that χ is distributed as the inverse gamma distribution with parameter $v_+ + v_-$, i.e., if $1/\chi$ obeys the gamma distribution with the same parameter. Its Nth moment is $1/\prod_{l=1}^{N}(v_+ + v_- - l)$ which compensates the extra factor.

Distributions

$$\boldsymbol{F(s) = \frac{1}{\kappa(\gamma_t^{-1}\frac{d}{ds})}\tilde{F}(s),}$$

where $\boldsymbol{\tilde{F}(s)} =$ Prob$[\boldsymbol{\tilde{h}(0,t) \leq \gamma_t s}]$, $\boldsymbol{F(s)} =$ Prob$[\boldsymbol{h(0,t) \leq \gamma_t s}]$ and $\boldsymbol{\kappa}$ is the Laplace transform of the pdf of $\boldsymbol{\chi}$. For the inverse gamma distribution, $\boldsymbol{\kappa(\xi) = \Gamma(v+\xi)/\Gamma(v)}$, by which we get the formula for the generating function.

Summary

- 1D KPZ equation is now under revival.
- Replica analysis is suitable for various generalizations.

  For KPZ replica analysis could be made rigorous.
- Explicit formulas for the stationary measure.

  Height distribution and two point correlation function.
- Generalization to ASEP? In Macdonald setting?