Ginibre point process and its Palm measures: absolute continuity and singularity

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1 Ginibre point process and determinantal point process

2 Main results

3 Absolute continuity

4 Singularity
Ginibre matrix ensemble and complex eigenvalues

- \( \mathcal{M}_N \): the space of \( N \times N \) complex matrices \( \cong \mathbb{C}^{N^2} \)
- \( P_N(dX) = Z_N^{-1} \exp(-\text{Tr}X^*X)dX \),

It is called the Ginibre matrix ensemble of size \( N \).

Probability density of complex eigenvalues was computed by Ginibre(1965) as follows:

\[
p_N(z_1; \ldots; z_N) = \frac{1}{\prod_{k=1}^N k!} \prod_{1 \leq i < j \leq N} |z_i - z_j|^2 \left( \det(z_j^i) \right)_{i, j=1}^N
\]
with respect to the standard complex Gaussian measure \( N(dz_1; \ldots; dz_N) \) with

\[
(dz) = \frac{1}{\pi} e^{-|z|^2} dz.
\]
Ginibre matrix ensemble and complex eigenvalues

- $\mathcal{M}_N$: the space of $N \times N$ complex matrices $\cong \mathbb{C}^{N^2}$
  
  $P_N(dX) = Z_N^{-1} \exp(-\text{Tr}X^*X)dX,$

- or equivalently,
  
  Random matrix whose entries are all i.i.d. standard complex Gaussian.
  It is called **Ginibre matrix ensemble** of size $N$. 

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Ginibre matrix ensemble and complex eigenvalues

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- Probability density of complex eigenvalues was computed by Ginibre(1965) as follows:

\[
p^{(N)}(z_1, \ldots, z_N) = \frac{1}{\prod_{k=1}^{N} k!} \prod_{1 \leq i < j \leq N} |z_i - z_j|^2
= \frac{1}{\prod_{k=1}^{N} k!} \det(z_i^{-1})_{i,j=1}^{N}
\]

with respect to the standard complex Gaussian measure $\lambda^\otimes N(dz_1 \ldots dz_N)$ with $\lambda(dz) = \pi^{-1} e^{-|z|^2} dz$. 

Figure: $N = 100, 400, 900$
Random complex eigenvalues

Bai showed that \( \frac{1}{N} \sum_{i=1}^{N} \delta_{z_i/\sqrt{N}} \xrightarrow{w} \text{Uniform}(D_1) \) almost surely
Definition (Determinantal point process (DPP))

A DPP is a point process having determinantal correlation functions

$$\rho_n(z_1, z_2, \ldots, z_n) = \det(K(z_i, z_j)_{i,j=1}^n)$$

for some $K(z, w)$ relative to a Radon measure $\lambda(dz) = g(z)dz$. 

The $N$-particle Ginibre point process on $\mathbb{C}$ is rotation invariant DPP on $\mathbb{C}$ whose kernel relative to $(dz) = \frac{1}{\pi} e^{-|z|^2} \, dz$ is given by

$$K(N)(z; w) = \frac{N!}{\prod_{k=0}^{N-1} (z - w)^k} e^{-|z|^2 - |w|^2} = K(z; w).$$

When correlation functions converge uniformly on any compacts, corresponding point processes converge weakly to a limit.
**Definition (Determinantal point process (DPP))**

DPP is a point process having determinantal correlation functions

\[ \rho_n(z_1, z_2, \ldots, z_n) = \det(K(z_i, z_j)_{i,j=1}^n) \]

for some \( K(z, w) \) relative to a Radon measure \( \lambda(dz) = g(z)dz \).

- The \( N \)-particle Ginibre point process on \( \mathbb{C} \) is rotation invariant DPP on \( \mathbb{C} \) whose kernel relative to \( \lambda(dz) = \pi^{-1} e^{-|z|^2} dz \) is given by

\[
K^{(N)}(z, w) = \sum_{k=0}^{N-1} \frac{(z \overline{w})^k}{k!}
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\[
K^{(N)}(z, w) = \sum_{k=0}^{N-1} \frac{(zw)^k}{k!} \quad N \to \infty \quad e^{z\bar{w}} =: K(z, w)
\]
Definition (Determinantal point process (DPP))

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- When correlation functions converge uniformly on any compacts, corresponding point processes converge weakly to a limit.
Ginibre point process

- **Ginibre point process** on \( \mathbb{C} \) is defined as DPP with a kernel

\[
K(z, w) = e^{z\bar{w}}, \quad \lambda(dz) = \pi^{-1} e^{-|z|^2} dz
\]
Ginibre point process on $\mathbb{C}$ is defined as DPP with a kernel

$$K(z, w) = e^{z\bar{w}}, \quad \lambda(dz) = \pi^{-1} e^{-|z|^2} dz$$

In particular,

$$\rho_1(z) = g(z)K(z, z) = \pi^{-1}$$
$$\rho_2(z, w) \leq \rho_1(z)\rho_1(w) \quad \cdots \text{negative correlation}$$
Ginibre point process

- **Ginibre point process** on $\mathbb{C}$ is defined as DPP with a kernel

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In particular,

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- Ginibre p.p. on $\mathbb{C}$ is invariant under translations and rotations.
Figure: Poisson(left) and Ginibre(right)
\( \xi(D_r) \) is the number of points inside the disk of radius \( r \).

1. **Variance**
   - Poisson case:
     \[
     \text{Var}(\xi(D_r)) = r^2
     \]
   - Ginibre case:
     \[
     \text{Var}(\xi(D_r)) \sim \frac{r}{\sqrt{\pi}}
     \]
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     \]
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     \[
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     \]

2. **Large deviations**
   - Poisson case:
     \[
     P(r^{-2}\xi(D_r) \approx a) \sim \exp(-I(a)r^2)
     \]
   - Ginibre case:
     \[
     P(r^{-2}\xi(D_r) \approx a) \sim \exp(-J(a)r^4)
     \]
Ginibre point process is considered as a Gibbs measure?

- Formal expression:

\[
\mu = Z^{-1} \prod_{i<j} |z_i - z_j|^2 e^{-\sum_i |z_i|^2} \prod_{i=1}^{\infty} dz_i
\]

\[
= Z^{-1} \exp \left( -\sum_i |z_i|^2 + 2 \prod_{i<j} \log |z_i - z_j| \right) \prod_{i=1}^{\infty} dz_i
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$$

- 2-body potential $\Phi(z, w) = -2 \log |z - w|$ is not even bounded.
Palm measure

- We focus on the (reduced) Palm measure of a simple point process $\mu$ defined as follows: for $a = (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$

$$\mu^a(\cdot) := \mu(\cdot - \sum_{i=1}^{n} \delta_{a_i} \mid \xi(\{a_i\}) \geq 1, \forall i = 1, 2, \ldots n)$$
Palm measure

- We focus on the (reduced) Palm measure of a simple point process $\mu$ defined as follows: for $a = (a_1, a_2, \ldots, a_n) \in R^n$

  $$\nu^a(\cdot) := \mu(\cdot - \sum_{i=1}^{n} \delta_{a_i} \mid \xi(\{a_i\}) \geq 1, \forall i = 1, 2, \ldots n)$$

- For a Poisson point process $\Pi$, it is well-known that

  $$\Pi^a = \Pi$$
Palm measure

- We focus on the (reduced) Palm measure of a simple point process $\mu$ defined as follows: for $a = (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$

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\mu^a(\cdot) := \mu(\cdot - \sum_{i=1}^{n} \delta_{a_i} | \xi(\{a_i\}) \geq 1, \forall i = 1, 2, \ldots n)
$$

- For a Poisson point process $\Pi$, it is well-known that

$$
\Pi^a = \Pi
$$

- More generally, for a Gibbs measure (with nice potential $U$), it is well-known that

$$
\frac{d\mu^a}{d\mu}(\xi) \propto e^{-U(a|\xi)}
$$

where $U(a|\xi)$ is the energy from the other configuration $\xi$. 
Palm measure of DPP is again a DPP and its kernel is given by

\[ K^\alpha(z, w) = K(z, w) - \frac{K(z, \alpha)K(\alpha, w)}{K(\alpha, \alpha)} \]
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For Ginibre point process \((K(z, w) = e^{\bar{z}w})\),

\[ K^0(z, w) = K(z, w) - \frac{K(z, 0)K(0, w)}{K(0, 0)} = e^{\bar{z}w} - 1 \]

with respect to \( \lambda(dz) = \pi^{-1}e^{-|z|^2}dz =: g(z)dz \).
Ginibre point process and its Palm measure

- Palm measure of DPP is again a DPP and its kernel is given by

\[ K^\alpha(z, w) = K(z, w) - \frac{K(z, \alpha)K(\alpha, w)}{K(\alpha, \alpha)} \]

- For Ginibre point process \((K(z, w) = e^{zw})\),

\[ K^0(z, w) = K(z, w) - \frac{K(z, 0)K(0, w)}{K(0, 0)} = e^{zw} - 1 \]

with respect to \(\lambda(dz) = \pi^{-1}e^{-|z|^2}dz =: g(z)dz\).

- In particular, the particle density under Palm measure is reduced to

\[ \rho^0_1(z) = g(z)K^0(z, z) = \pi^{-1}(1 - e^{-|z|^2}). \]
Questions

1. Are Ginibre p.p. $\mu$ and its Palm measure $\mu^a$ absolutely continuous?
2. Are Palm measures of Ginibre p.p. $\mu^a$ and $\mu^b$ absolutely continuous?
3. Give a criterion for absolute continuity between general DPPs in terms of kernel $K$ and base measure $\lambda$. 
Theorem (A.V. Skorohod, Y. Takahashi)

Let $\Pi_\lambda$ be a Poisson point process with intensity $\lambda$. Then, $\Pi_\lambda \sim \Pi_\rho$ are equivalent to the following:

(i) $\lambda \sim \rho$

(ii) Hellinger distance between $\lambda$ and $\rho$ is finite

\[
d(\lambda, \rho)^2 = \frac{1}{2} \int_R \left( \sqrt{\frac{d\rho}{d\lambda}} - 1 \right)^2 d\lambda < \infty
\]

Moreover,

\[
D(\Pi_\lambda, \Pi_\rho)^2 := \frac{1}{2} \int_R \left( \sqrt{\frac{d\Pi_\rho}{d\Pi_\lambda}} - 1 \right)^2 d\Pi_\lambda = 1 - e^{-d(\lambda, \rho)^2}
\]
Main results

- $\mu$: Ginibre point process on $\mathbb{C}$
- $\mu^x$: the Palm measure of $\mu$ given that there are points at $x \in \mathbb{C}^m$.

Theorem

Let $x \in \mathbb{C}^m$ and $y \in \mathbb{C}^n$, where $m, n = 0, 1, 2, \ldots$. Then the following holds.

(i) If $m = n$, then $\mu^x$ and $\mu^y$ are mutually absolutely continuous.

(ii) If $m \neq n$, then $\mu^x$ and $\mu^y$ are singular each other.

When $\mu$ is Gibbs, $x \ll y$. So, we would say that Ginibre is not Gibbs in the ordinary sense. Osada introduced a weak notion of Gibbs measure, quasi-Gibbs property.
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When $\mu$ is Gibbs, $\mu^x \ll \mu$. So, we would say that Ginibre is not Gibbs in the ordinary sense. Osada introduced a weak notion of Gibbs measure, quasi-Gibbs property.
Consider the zeros of the (hyperbolic) Gaussian analytic function

\[ X(z) = \sum_{n=0}^{\infty} \zeta_n z^n \quad \text{on } D_1 \]

where \( \zeta_n, n = 0, 1, 2, \ldots \) i.i.d. \( \mathcal{N}_\mathbb{C}(0, 1) \).

Peres-Virág showed that zeros of \( X(z) \) form DPP associated with Bergman kernel \( K(z, w) = \pi^{-1}(1 - zw)^{-2} \) and Lebesgue measure on \( D_1 \).

Halroyd-Soo showed that the zero process \( \mu_X \) and its Palm measure \( \mu_X^0 \) are mutually absolutely continuous.
Absolute continuity

**Theorem**

For any \( x, y \in \mathbb{C}^n \), two Palm measures \( \mu^x \) and \( \mu^y \) of the Ginibre point process are mutually absolutely continuous and its Radon-Nikodym density is given by

\[
\frac{d\mu^x}{d\mu^y}(\xi) = \frac{1}{Z_{xy}} \lim_{r \to \infty} \prod_{|z_i| < br} \frac{|x - z_i|^2}{|y - z_i|^2}
\]

where \( \xi = \sum_i \delta_{z_i} \), \( |x - z|^2 = \prod_{j=1}^n |x_j - z|^2 \) and

\[
Z_{xy} = \frac{\det(K(x_i, x_j))_{i,j=1}^n}{\det(K(y_i, y_j))_{i,j=1}^n}
\]

The infinite product of the RHS is conditionally convergent.
Radon-Nikodym density

For simplicity, $x \in \mathbb{C}$ and $\xi = \sum_i \delta_{z_i}$

- $\mu^x(z_1, \ldots, z_n) \propto \prod_{i=1}^n |x - z_i|^2 \prod_{i < j} |z_i - z_j|^2 \exp(-\sum_{i=1}^n |z_i|^2)$
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- Radon-Nikodym density is

$$\frac{\mu^x(z_1, \ldots, z_n)}{\mu^y(z_1, \ldots, z_n)} = \prod_{i=1}^n \frac{|x - z_i|^2}{|y - z_i|^2} = \prod_{i=1}^n \frac{|1 - x/z_i|^2}{|1 - y/z_i|^2} \to ?$$
Radon-Nikodym density

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\frac{\mu^x(z_1, \ldots, z_n)}{\mu^y(z_1, \ldots, z_n)} = \prod_{i=1}^n \frac{|x - z_i|^2}{|y - z_i|^2} = \prod_{i=1}^n \frac{|1 - x/z_i|^2}{|1 - y/z_i|^2} \rightarrow ?
\]

- The canonical infinite product (of order 2)

\[
\prod_{i=1}^\infty \left(1 - \frac{x}{z_i}\right) \exp \left(\frac{x}{z_i} + \frac{x^2}{2z_i^2}\right)
\]

is absolutely convergent, but $\prod_{i=1}^\infty \left(1 - \frac{x}{z_i}\right)$ itself is not since the number of zeros $\xi(\mathbb{D}_r)$ inside the disk $\mathbb{D}_r$ grows like $r^2$. 
Small fluctuation

- $\xi(D_r)$ is the number of points inside the disk of radius $r$. 

\[ \text{var}(\xi(D_r)) = O(r^2) \text{ under Poisson p.p.,} \]
\[ \text{var}(\xi(D_r)) = O(r) \text{ as } r \to 1 \text{ under Ginibre p.p.} \]
Small fluctuation

- $\xi(D_r)$ is the number of points inside the disk of radius $r$.
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$$\text{var}(\xi(D_r)) = O(r) \quad \text{as } r \to \infty$$

- This small fluctuation property makes the series

$$\sum_{|z_i|<br} \frac{1}{z_i}, \quad \sum_{|z_i|<br} \frac{1}{z_i^2}$$

to be conditionally convergent, and hence $\prod_{i=1}^{\infty} \left(1 - \frac{x}{z_i}\right)$ is conditionally convergent.
Small fluctuation

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- This small fluctuation property makes the series $\sum \frac{1}{z_i}$, $\sum \frac{1}{z_i^2}$ to be conditionally convergent, and hence $\prod_{i=1}^{\infty} (1 - \frac{x}{z_i})$ is conditionally convergent.

- The situation is similar to

$$\frac{\sin \pi Z}{\pi Z} = \prod_{i \in \mathbb{Z} \setminus \{0\}} (1 - \frac{Z}{i}) = \prod_{i=1}^{\infty} (1 - \frac{Z}{i})(1 - \frac{Z}{-i}) = \prod_{i=1}^{\infty} (1 - \frac{Z^2}{i^2})$$
Katori-Tanemura and Osada independently (by using different techniques) constructed diffusion processes of $\infty$-particles invariant under some DPPs on $\mathbb{R}$.

Osada also constructed a diffusion process invariant under Ginibre point process. It is formally given as $\infty$-dimensional SDE

$$dX^i_t = dB^i_t - X^i_t + \lim_{r \to \infty} \sum_{|X^i_t| < r, j \neq i} \frac{X^i_t - X^j_t}{|X^i_t - X^j_t|^2} dt$$
For singularity: Kostlan’s theorem

- $Y_1, Y_2, \ldots$ are independent and $Y_i \sim \Gamma(i, 1)$, the sum of $i$ exponential random variables with mean 1.

$$\{ |z_1|^2, |z_2|^2, \ldots \} \overset{d}{=} \{ Y_1, Y_2, \ldots \}$$ as a set

where $\xi = \sum_i \delta_{z_i}$ is a Ginibre point configuration.
For singularity: Kostlan’s theorem

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- Under the Palm measure conditioned at the origin, we see that
  
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\]

- Kakutani’s dichotomy for two independent infinite sequences of probability measures \( \mathcal{M} = (\mu_1, \mu_2, \ldots) \), and \( \mathcal{N} = (\nu_1, \nu_2, \ldots) \),

\[
\prod_{i=1}^{\infty} \int_{\mathbb{R}} \sqrt{\mu_i(t)\nu_i(t)}dt > 0 \text{ or } = 0 \iff \mathcal{M} \sim \mathcal{N} \text{ or } \mathcal{M} \perp \mathcal{N}
\]
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- $Y_1, Y_2, \ldots$ are independent and $Y_i \sim \Gamma(i, 1)$, the sum of $i$ exponential random variables with mean 1.

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- By some calculation, we see that, $(Y_1, Y_2, \ldots) \perp (Y_2, Y_3, \ldots)$. 

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The zeros of Gaussian analytic function

- \( GAF \, X(z) = \sum_{n=0}^{\infty} \zeta_n z^n \), where \( \zeta_n \sim N_\mathbb{C}(0, 1) \)
- \( Y_1, Y_2, \ldots \) are independent and \( Y_i \sim U_i^{1/2i} \).

\[ \{|z_1|^2, |z_2|^2, \ldots\} \overset{d}{=} \{Y_1, Y_2, \ldots\} \quad \text{as a set} \]

where \( \xi = \sum_i \delta_{z_i} \) is the zeros of the above GAF \( X \).

- Under the Palm measure conditioned at the origin, we see that
  \[ \{|z_1|^2, |z_2|^2, \ldots\} \overset{d}{=} \{Y_2, Y_3, \ldots\} \quad \text{as a set} \]

- By some calculation, we see that
  \[ (Y_1, Y_2, \ldots) \sim (Y_2, Y_3, \ldots). \]
Image measures

- Image measures inherit absolute continuity but not necessarily singularity.
Image measures

- Image measures inherit absolute continuity but not necessarily singularity.
- Our situation is like this.
  The RHSs have the same distribution by Kostlan’s theorem.

\[
\prod_{i=1}^{\infty} \mathbb{C} \ni (Y_1, Y_2, \ldots) \mapsto \sum_{i=1}^{\infty} \delta_{Y_i} \in \text{Conf}([0, \infty))
\]

\[
\text{Conf}(\mathbb{C}) \ni \sum_{i=1}^{\infty} \delta_{z_i} \mapsto \sum_{i=1}^{\infty} \delta_{|z_i|^2} \in \text{Conf}([0, \infty))
\]
Idea of the proof for singularity (1)

- Define a function on the configuration space

\[ F_N(\xi) := \frac{1}{N} \sum_{k=1}^{N} (\xi(D_k) - k) \quad \text{for} \quad \xi = \sum_i \delta_{z_i} \]

where \( \xi(D_k) \) is the number of points inside the disk \( D_k \) of radius \( \sqrt{k} \).
Idea of the proof for singularity (1)

- Define a function on the configuration space

$$F_N(\xi) := \frac{1}{N} \sum_{k=1}^{N} (\xi(D_k) - k) \quad \text{for} \quad \xi = \sum_i \delta_{z_i}$$

where $\xi(D_k)$ is the number of point inside the disk $D_k$ of radius $\sqrt{k}$.

- Since $\xi(D_k), k = 1, 2, \ldots, n$ are correlated,

$$\text{var}(F_N) = O(N)$$

under Poisson p.p. $\Pi$. 
Idea of the proof for singularity (1)

- Define a function on the configuration space

\[ F_N(\xi) := \frac{1}{N} \sum_{k=1}^{N} (\xi(D_k) - k) \quad \text{for} \quad \xi = \sum_i \delta_{z_i} \]

where \( \xi(D_k) \) is the number of points inside the disk \( D_k \) of radius \( \sqrt{k} \).

- Since \( \xi(D_k), k = 1, 2, \ldots, n \) are correlated,

\[ \text{var}(F_N) = O(N) \]

under Poisson p.p. \( \Pi \).

Proposition

Under Ginibre p.p. and its Palm measures,

\[ \text{var}(F_N) = O(1) \]

by negative correlation.
Idea of the proof for singularity (2)

- $F_N(\xi) := \frac{1}{N} \sum_{k=1}^{N} (\xi(D_k) - k)$ for $\xi = \sum_i \delta_{z_i}$

Theorem

For $m \in \mathbb{N}$ and $0_m = (0, 0, \ldots, 0) \in \mathbb{C}^m$, then

$$\lim_{N \to \infty} F_N(\xi) = -m, \quad \text{weakly in } L^2(\mu 0_m)$$
Idea of the proof for singularity (2)

- \( F_N(\xi) := \frac{1}{N} \sum_{k=1}^{N} (\xi(D_k) - k) \) for \( \xi = \sum_i \delta_{z_i} \)

**Theorem**

*For* \( m \in \mathbb{N} \) *and* \( 0_m = (0, 0, \ldots, 0) \in \mathbb{C}^m \), *then*

\[
\lim_{N \to \infty} F_N(\xi) = -m, \quad \text{weakly in } L^2(\mu^{0_m})
\]

- From this theorem, we see that for \( 0_n \in \mathbb{C}^n \) *and* \( 0_m \in \mathbb{C}^m \) *with* \( n \neq m \), \( \mu^{0_n} \) *and* \( \mu^{0_m} \) *are mutually singular*, \( \mu^{0_n} \perp \mu^{0_m} \).
Idea of the proof for singularity (2)

- \( F_N(\xi) := \frac{1}{N} \sum_{k=1}^{N} (\xi(D_k) - k) \) for \( \xi = \sum_i \delta_{z_i} \)

**Theorem**

For \( m \in \mathbb{N} \) and \( 0_m = (0, 0, \ldots, 0) \in \mathbb{C}^m \), then

\[
\lim_{N \to \infty} F_N(\xi) = -m, \text{ weakly in } L^2(\mu_0^m)
\]

- From this theorem, we see that for \( 0_n \in \mathbb{C}^n \) and \( 0_m \in \mathbb{C}^m \) with \( n \neq m \), \( \mu_0^n \) and \( \mu_0^m \) are mutually singular, \( \mu_0^n \perp \mu_0^m \).
- For general \( a \in \mathbb{C}^n \) and \( b \in \mathbb{C}^m \),

\[
\mu_a \sim \mu_0^n \perp \mu_0^m \sim \mu_b
\]
Concluding remarks and open questions

1. Absolute continuity for general DPP. Can we give a criterion for absolute continuity in terms of $K(z, w)$ and $\lambda(dz)$? More concretely, radially symmetric DPP on $\mathbb{C}$ may be next target, for example.

2. Is it true that singularity are inherited via two mappings:

$$\prod_{i=1}^{\infty} \mathbb{C} \ni (Y_1, Y_2, \ldots) \mapsto \sum_{i=1}^{\infty} \delta_{Y_i} \in \text{Conf}([0, \infty))$$

$$\text{Conf}(\mathbb{C}) \ni \sum_{i=1}^{\infty} \delta_{z_i} \mapsto \sum_{i=1}^{\infty} \delta_{|z_i|^2} \in \text{Conf}([0, \infty))$$

3. Point processes on $\mathbb{C}$ defines probability measures on entire functions by Hadamard product. How does a point process affect random entire function? Can we say something about absolute continuity from properties of random entire functions obtained from point processes?
舟木さん、
還暦おめでとうございます。