Ginibre point process and its Palm measures: absolute continuity and singularity

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[1] Joint work with Hirofumi Osada (Kyushu University)

[2] "10th workshop on stochastic analysis on large scale interacting systems" at Kochi University, Dec. 5–7, 2011.

Contents

Ginibre matrix ensemble and complex eigenvalues

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  Random matrix whose entries are all i.i.d. standard complex Gaussian.
  It is called **Ginibre matrix ensemble** of size N.
- Probability density of complex eigenvalues was computed by Ginibre(1965) as follows:

$$p^{(N)}(z_1, \ldots, z_N) = \frac{1}{\prod_{k=1}^N k!} \prod_{1 \le i < j \le N} |z_i - z_j|^2$$

$$= \frac{1}{\prod_{k=1}^N k!} \det(z_i^{j-1})_{i,j=1}^N$$

with respect to the standard complex Gaussian measure $\lambda^{\otimes N}(dz_1 \ldots dz_N)$ with $\lambda(dz) = \pi^{-1} e^{-|z|^2} dz$.

Random complex eigenvalues

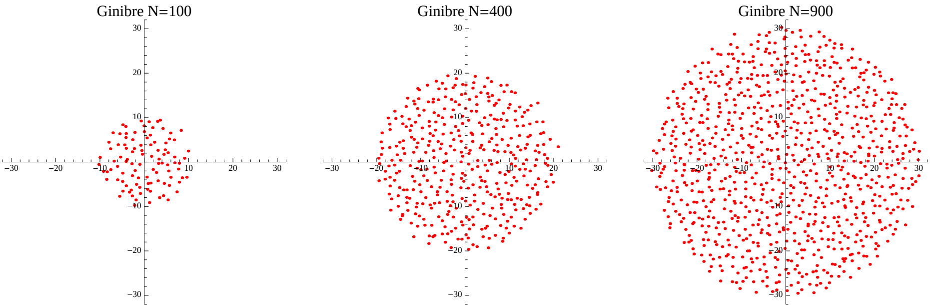
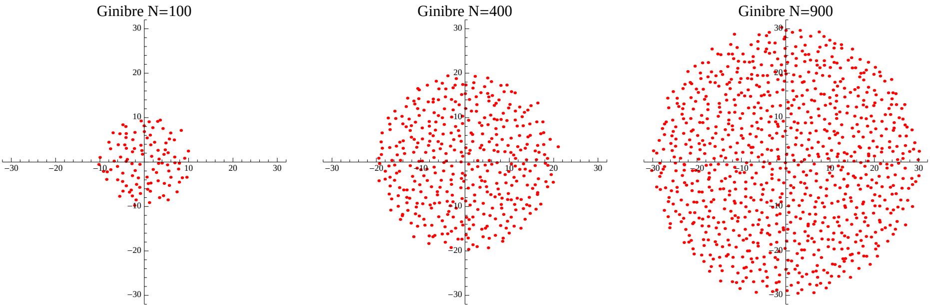


Figure: $N = 100, 400, 900$

Random complex eigenvalues



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- Bai showed that $\frac{1}{N}\sum_{i=1}^{N} \delta_{z_i/\sqrt{N}} \xrightarrow{w} Uniform(D_1)$ almost surely

Ginibre point process as determinantal point process

Definition (Determinantal point process (DPP))

DPP is a point process having deteminantal correlation functions

$$\rho_n(z_1, z_2, \dots, z_n) = \det(K(z_i, z_j))_{i,j=1}^n$$

for some $K(z, w)$ *relative to a Radon measure* $\lambda(dz) = g(z)dz$.

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- The N-particle Ginibre point process on $\mathbb{C}$ is rotation invariant DPP on $\mathbb{C}$ whose kernel relative to $\lambda(dz) = \pi^{-1} e^{-|z|^2} dz$ is given by

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- When correlation functions converge uniformly on any compacts, corresponding point processes converge weakly to a limit.

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In particular,

$$\rho_1(z) = g(z)K(z,z) = \pi^{-1}$$
$$\rho_2(z,w) \le \rho_1(z)\rho_1(w) \quad \cdots \text{ negative correlation}$$

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- Ginibre p.p. on $\mathbb{C}$ is invariant under translations and rotations.

Poisson and Ginibre

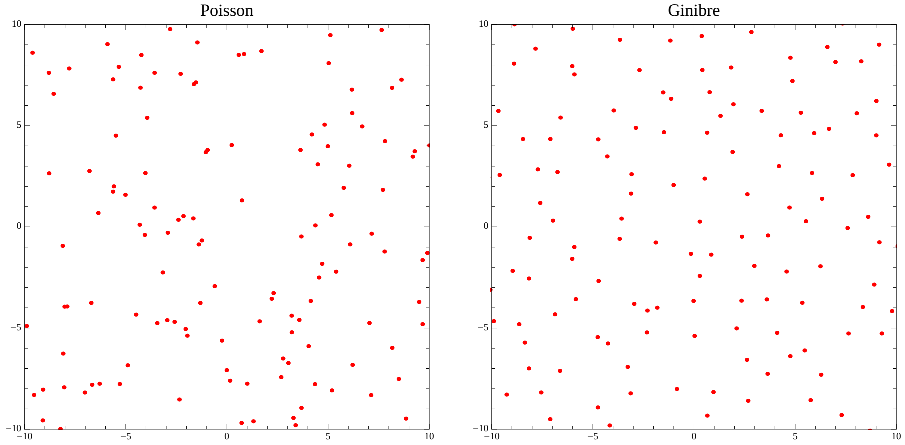

Figure: Poisson(left) and Ginibre(right)

Variance and large deviations for the number of points

- $\xi(D_r)$ is the number of points inside the disk of radius r.

1. Variance
   - Poisson case:
   $$Var(\xi(D_r)) = r^2$$
   - Ginibre case:
   $$Var(\xi(D_r)) \sim \frac{r}{\sqrt{\pi}}$$

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2. Large deviations
   - Poisson case:
$$P(r^{-2}\xi(D_r) \approx a) \sim \exp(-I(a)r^2)$$
   - Ginibre case:
$$P(r^{-2}\xi(D_r) \approx a) \sim \exp(-J(a)r^4)$$

Ginibre point process is considered as a Gibbs measure?

- Formal expression:

$$\mu = Z^{-1} \prod_{i<j} |z_i - z_j|^2 e^{-\sum_i |z_i|^2} \prod_{i=1}^{\infty} dz_i$$

$$= Z^{-1} \exp\left(-\sum_i |z_i|^2 + 2 \prod_{i<j} \log |z_i - z_j| \right) \prod_{i=1}^{\infty} dz_i$$

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- 2-body potential $\Phi(z, w) = -2 \log |z - w|$ is not even bounded.

Palm measure

- We focus on the (reduced) Palm measure of a simple point process μ defined as follows: for $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in R^n$

$$\mu^{\mathbf{a}}(\cdot) := \mu(\cdot - \sum_{i=1}^{n} \delta_{a_i} \mid \xi(\{a_i\}) \geq 1, \forall i = 1, 2, \ldots n)$$

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- More generally, for a Gibbs measure (with nice potential U), it is well-known that

$$\frac{d\mu^{\mathbf{a}}}{d\mu}(\xi) \propto e^{-U(\mathbf{a}|\xi)}$$

where $U(\mathbf{a}|\xi)$ is the energy from the other configuration ξ.

Ginibre point process and its Palm measure

- Palm measure of DPP is again a DPP and its kernel is given by

$$K^{\alpha}(z,w) = K(z,w) - \frac{K(z,\alpha)K(\alpha,w)}{K(\alpha,\alpha)}$$

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- For Ginibre point process ($K(z,w) = e^{z\overline{w}}$),

$$K^{0}(z,w) = K(z,w) - \frac{K(z,0)K(0,w)}{K(0,0)} = e^{z\overline{w}} - 1$$

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  with respect to $\lambda(dz) = \pi^{-1}e^{-|z|^2}dz =: g(z)dz$.
- In particular, the particle density under Palm measure is reduced to

$$\rho_1^0(z) = g(z)K^0(z,z) = \pi^{-1}(1 - e^{-|z|^2}).$$

Questions

1. Are Ginibre p.p. μ and its Palm measure $\mu^{\mathbf{a}}$ absolutely continuous?
2. Are Palm measures of Ginibre p.p. $\mu^{\mathbf{a}}$ and $\mu^{\mathbf{b}}$ absolutely continuous?
3. Give a criterion for absolute continuity between general DPPs in terms of kernel K and base measure λ.

Absolute continuity for Poisson p.p

Theorem (A.V.Skorohod, Y.Takahashi)

Let Π_λ be a Poisson point process with intensity λ. Then, $\Pi_\lambda \sim \Pi_\rho$ are equivalent to the following:
(i) $\lambda \sim \rho$
(ii) Hellinger distance between λ and ρ is finite

$$d(\lambda, \rho)^2 = \frac{1}{2} \int_R \left(\sqrt{\frac{d\rho}{d\lambda}} - 1 \right)^2 d\lambda < \infty$$

Moreover,

$$D(\Pi_\lambda, \Pi_\rho)^2 := \frac{1}{2} \int_R \left(\sqrt{\frac{d\Pi_\rho}{d\Pi_\lambda}} - 1 \right)^2 d\Pi_\lambda = 1 - e^{-d(\lambda,\rho)^2}$$

Main results

- μ : Ginibre point process on $\mathbb{C}$
- $\mu^{\mathbf{x}}$: the Palm measure of μ given that there are points at $\mathbf{x} \in \mathbb{C}^m$.

Theorem

Let $\mathbf{x} \in \mathbb{C}^m$ and $\mathbf{y} \in \mathbb{C}^n$, where $m, n = 0, 1, 2, \ldots$. Then the following holds.

(i) If $m = n$, then $\mu^{\mathbf{x}}$ and $\mu^{\mathbf{y}}$ are mutually absolutely continuous.

(ii) If $m \neq n$, then $\mu^{\mathbf{x}}$ and $\mu^{\mathbf{y}}$ are sigular each other.

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- When μ is Gibbs, $\mu^{\mathbf{x}} << \mu$. So, we would say that Ginibre is not Gibbs in the ordinary sense. Osada introduced a weak notion of Gibbs measure, quasi-Gibbs property.

Related results

- Consider the zeros of the (hyperbolic) Gaussian analytic function

$$X(z) = \sum_{n=0}^{\infty} \zeta_n z^n \quad \text{on } D_1$$

where $\zeta_n, n = 0, 1, 2, \dots$ i.i.d. $N_{\mathbb{C}}(0, 1)$.

- Peres-Virág showed that zeros of $X(z)$ form DPP associated with Bergman kernel $K(z, w) = \pi^{-1}(1 - z\overline{w})^{-2}$ and Lebesgue measure on D_1.
- Halroyd-Soo showed that the zero process μ_X and its Palm measure μ_X^0 are mutually absolutely continuous.

Absolute continuity

Theorem

For any $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$, *two Palm measures* $\mu^{\mathbf{x}}$ *and* $\mu^{\mathbf{y}}$ *of the Ginibre point process are mutually absolutely continuous and its Radon-Nikodym density is given by*

$$\frac{d\mu^{\mathbf{x}}}{d\mu^{\mathbf{y}}}(\xi) = \frac{1}{Z_{\mathbf{xy}}} \lim_{r\to\infty} \prod_{|z_i|<b_r} \frac{|\mathbf{x} - z_i|^2}{|\mathbf{y} - z_i|^2}$$

where $\xi = \sum_i \delta_{z_i}$, $|\mathbf{x} - z|^2 = \prod_{j=1}^n |x_j - z|^2$ *and*

$$Z_{\mathbf{xy}} = \frac{\det(K(x_i, x_j))_{i,j=1}^n}{\det(K(y_i, y_j))_{i,j=1}^n}$$

The infinite product of the RHS is conditionally convergent.

Radon-Nikodym density

For simplicity, $x \in \mathbb{C}$ and $\xi = \sum_i \delta_{z_i}$

- $\mu^x(z_1, \dots, z_n) \propto \prod_{i=1}^n |x - z_i|^2 \prod_{i<j} |z_i - z_j|^2 \exp(-\sum_{i=1}^n |z_i|^2)$

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- Radon-Nikodym density is

$$\frac{\mu^x(z_1, \ldots, z_n)}{\mu^y(z_1, \ldots, z_n)} = \prod_{i=1}^n \frac{|x - z_i|^2}{|y - z_i|^2} = \prod_{i=1}^n \frac{|1 - x/z_i|^2}{|1 - y/z_i|^2} \to ?$$

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- The canonical infinite product (of order 2)

$$\prod_{i=1}^{\infty} (1 - \frac{x}{z_i}) \exp\left(\frac{x}{z_i} + \frac{x^2}{2z_i^2}\right)$$

is absolutely covergent, but $\prod_{i=1}^{\infty}(1 - \frac{x}{z_i})$ itself is not since the number of zeros $\xi(D_r)$ inside the disk D_r grows like r^2.

Small fluctuation

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  under Ginibre p.p.
- This small fluctuation property makes the series $\displaystyle\sum_{|z_i|<b_r} \frac{1}{z_i}, \quad \sum_{|z_i|<b_r} \frac{1}{z_i^2}$ to be conditionally convergent, and hence $\prod_{i=1}^{\infty}(1 - \frac{x}{z_i})$ is conditinally convergent.

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- The situation is similar to

$$\frac{\sin \pi z}{\pi z} = \prod_{i \in \mathbb{Z}\setminus\{0\}} (1 - \frac{z}{i}) = \prod_{i=1}^{\infty}(1 - \frac{z}{i})(1 - \frac{z}{-i}) = \prod_{i=1}^{\infty}(1 - \frac{z^2}{i^2})$$

Interacting Brownian Motion under Ginibre p.p.

- Katori-Tanemura and Osada independently (by using different techniques) constructed diffusion processes of ∞-particles invariant under some DPPs on $\mathbb{R}$.
- Osada also constructed a diffusion process invariant under Ginibre point process. It is formally given as ∞-dimensional SDE

$$dX_t^i = dB_t^i - X_t^i + \lim_{r\to\infty} \sum_{|X_t^i|<r, j\neq i} \frac{X_t^i - X_t^j}{|X_t^i - X_t^j|^2} dt$$

For singularity: Kostlan's theorem

- $Y_1, Y_2, \dots$ are indepedent and $Y_i \sim \Gamma(i, 1)$, the sum of i exponential random variables with mean 1.

$$\{|z_1|^2, |z_2|^2, \dots\} \stackrel{d}{=} \{Y_1, Y_2, \dots\} \quad \text{as a set}$$

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- Under the Palm measure conditioned at the origin, we see that

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- Kakutani's dichotomy for two independent infinite sequences of probability measures $\mathcal{M} = (\mu_1, \mu_2, \dots)$, and $\mathcal{N} = (\nu_1, \nu_2, \dots)$,

$$\prod_{i=1}^{\infty} \int_{\mathbb{R}} \sqrt{\mu_i(t)\nu_i(t)}dt > 0 \text{ or } = 0 \Longleftrightarrow \mathcal{M} \sim \mathcal{N} \text{ or } \mathcal{M} \perp \mathcal{N}$$

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- By some calculation, we see that, $(Y_1, Y_2, \dots) \perp (Y_2, Y_3, \dots)$.

The zeros of Gaussian analytic function

- GAF $X(z) = \sum_{n=0}^{\infty} \zeta_n z^n$, where $\zeta_n \sim N_{\mathbb{C}}(0,1)$
- $Y_1, Y_2, \ldots$ are indepedent and $Y_i \sim U_i^{1/2i}$.

$$\{|z_1|^2, |z_2|^2, \ldots\} \stackrel{d}{=} \{Y_1, Y_2, \ldots\} \quad \text{as a set}$$

  where $\xi = \sum_i \delta_{z_i}$ is the zeros of the above GAF X.
- Under the Palm measure conditioned at the origin, we see that

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Image measures

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- Our situation is like this.
  The RHSs have the same distribution by Kostlan's theorem.

$$\prod_{i=1}^{\infty} \mathbb{C} \ni (Y_1, Y_2, \dots) \mapsto \sum_{i=1}^{\infty} \delta_{Y_i} \in Conf([0,\infty))$$

$$Conf(\mathbb{C}) \ni \sum_{i=1}^{\infty} \delta_{z_i} \mapsto \sum_{i=1}^{\infty} \delta_{|z_i|^2} \in Conf([0,\infty))$$

Idea of the proof for singularity (1)

- Define a function on the configuration space

$$F_N(\xi) := \frac{1}{N}\sum_{k=1}^{N}(\xi(D_k) - k) \quad \text{for } \xi = \sum_i \delta_{z_i}$$

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Proposition

Under Ginibre p.p. and its Palm measures,

$$\mathrm{var}(F_N) = O(1)$$

by negative correlation.

Idea of the proof for singularity (2)

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Theorem

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- For general $\mathbf{a} \in \mathbb{C}^n$ and $\mathbf{b} \in \mathbb{C}^m$,

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Concluding remarks and open questions

1. Absolute continuity for general DPP. Can we give a criterion for absolute continuity in terms of $K(z, w)$ and $\lambda(dz)$? More concretely, radially symmetric DPP on $\mathbb{C}$ may be next target, for example.
2. Is it true that singularity are inherited via two mappings:

$$\prod_{i=1}^{\infty} \mathbb{C} \ni (Y_1, Y_2, \dots) \mapsto \sum_{i=1}^{\infty} \delta_{Y_i} \in Conf([0, \infty))$$

$$Conf(\mathbb{C}) \ni \sum_{i=1}^{\infty} \delta_{z_i} \mapsto \sum_{i=1}^{\infty} \delta_{|z_i|^2} \in Conf([0, \infty))$$

3. Point processes on $\mathbb{C}$ defines probability measures on entire functions by Hadamard product. How does a point process affect random entire function? Can we say something about absolute continuity from properties of random entire functions obtained from point processes?

舟木さん，：
還暦おめでとうございます．