Kochi 07/12/2011

Discretizing the One-dimensional KPZ Equation

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- KPZ equation / exact solution
- lattice discretization of KPZ

jointly with T. Sasamoto (2009)

- lattice discretization of SHE

jointly with S. Prolhac (2011)

1. KPZ equation / exact solution   Kardar, Parisi, Zhang (1986)

height function $h(t): \mathbb{R} \to \mathbb{R}, \; t \geq 0$

$$\frac{\partial}{\partial t} h = \frac{1}{2}\left(\frac{\partial}{\partial x} h\right)^2 + \frac{1}{2}\frac{\partial^2}{\partial x^2} h + W$$

- $W(x,t)$ Gaussian white noise $\langle \cdot \rangle$
- Cauchy problem, initial data $h(x,0)$ could be random

⇒ Cole-Hopf transform

$$Z(x,t) = e^{h(x,t)}$$

$$\frac{\partial}{\partial t} Z = \frac{1}{2}\frac{\partial^2}{\partial x^2} Z + WZ$$

// stochastic heat equation //

Ito

auxiliary Brownian motion $b(t)$ "directed polymer"

$$Z(x,t) = \mathbb{E}_x\left(\exp\left[\int_0^t W(b(s), t-s)\, ds \right] Z(b(t), 0) \right)$$

// random partition function //

$h = \log Z$ // random free energy //

$Z > 0$, $\langle Z(x,t)^2 \rangle < \infty$ Bertini, Giacomin (1995)

$Z(x,t)$ Hölder $\frac{1}{2}$ in x, Hölder $\frac{1}{2}$ in t

⇝ Cole-Hopf solution $h = \log Z$

Hairer (2011) solution theory for KPZ, method of rough paths

- can be used for

1) $\frac{1}{2}\left(\frac{\partial h}{\partial x}\right)^2 + \frac{1}{4}\left(\frac{\partial h}{\partial x}\right)^4$ renormalizes to $\lambda\left(\frac{\partial h}{\partial x}\right)^2$

2) vector-valued KPZ ‖ 1D nonlinear fluctuating hydrodynamics
van Beijeren 2011

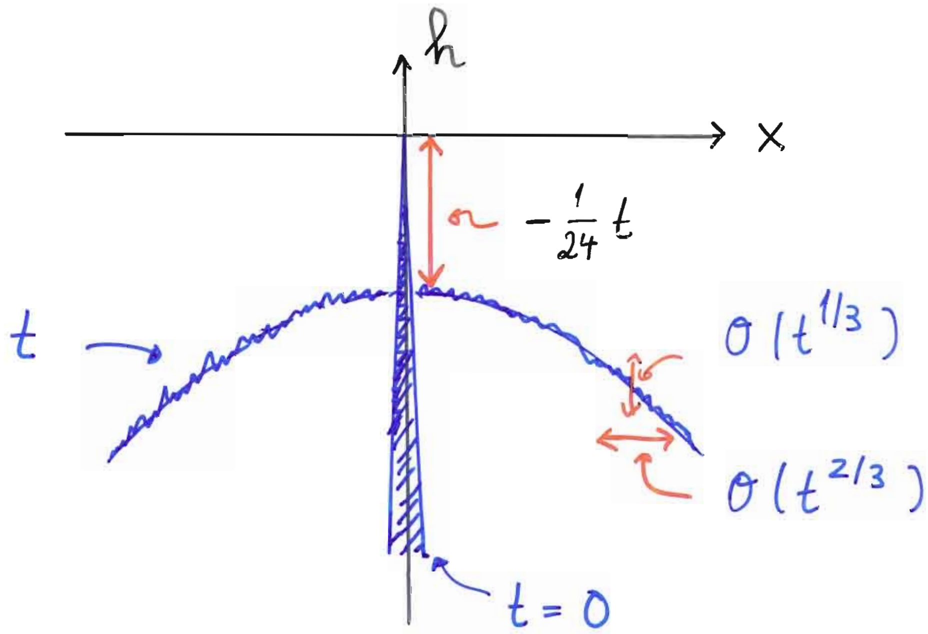
$$\frac{\partial}{\partial t} h_\alpha = c_\alpha \frac{\partial}{\partial x} h_\alpha + \frac{1}{2}\sum_{\beta,\gamma} b_\alpha^{\beta\gamma} \frac{\partial}{\partial x} h_\beta \frac{\partial}{\partial x} h_\gamma + \underbrace{\frac{1}{2}\frac{\partial^2}{\partial x^2} h_\alpha + W_\alpha}_{\text{"more general"}}$$

Cole-Hopf solution is NOT available

exact solution

sharp wedge initial data

$$Z(x,0) = \delta(x)$$



$$h(0,t) = -\frac{1}{24}t + \left(\frac{1}{2}t\right)^{1/3}\xi(t)$$

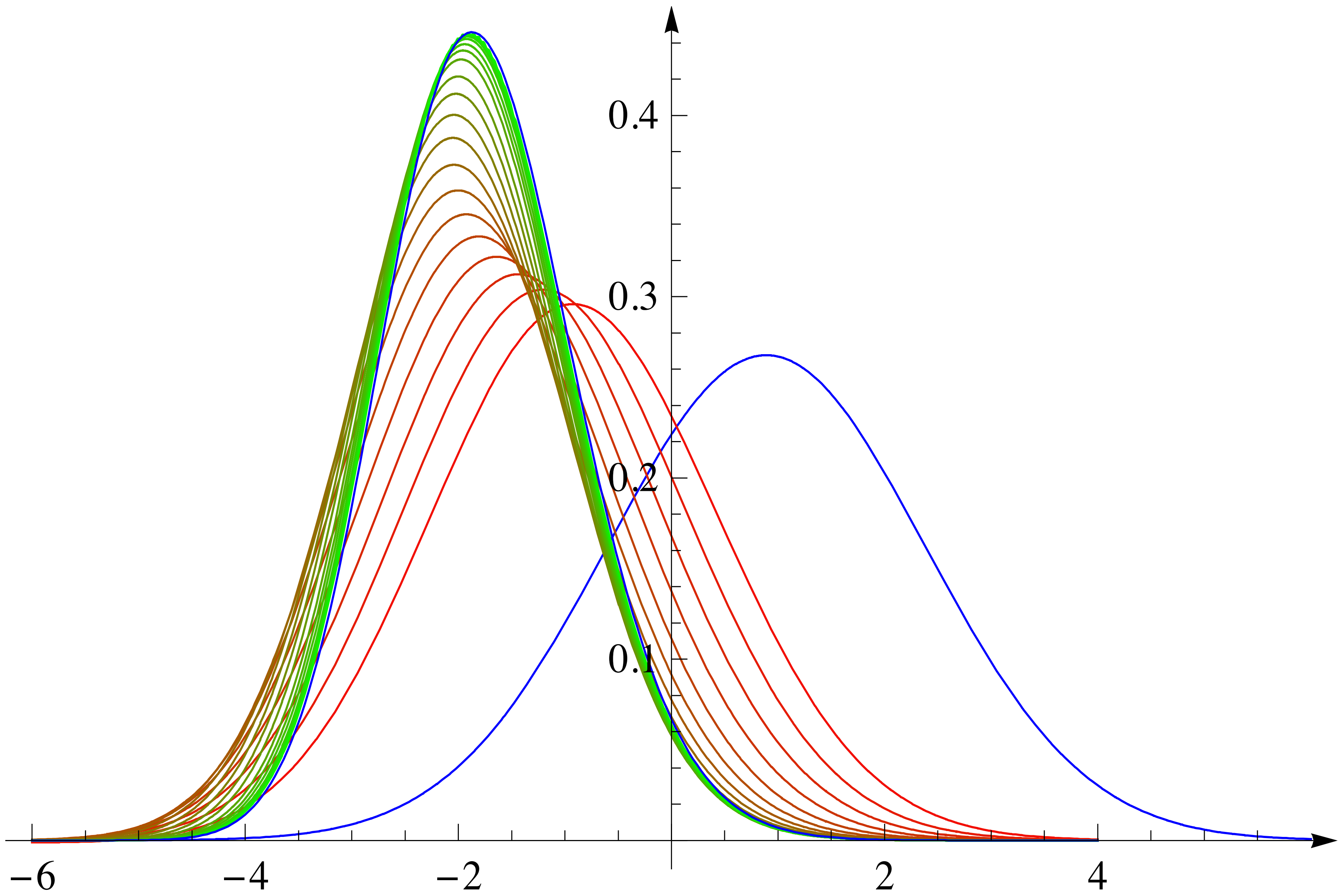
$\mathbb{P}(\xi(t) \in du) =$ Fredholm determinant on $L^2(\mathbb{R})$

fixed t : $x \mapsto h(x,t) + \frac{1}{2t}x^2$

stationary in x

plot: S. Prolhac (2011)

based on the method of Bornemann (2008)



0.4
0.3
0.2
0.1
−6
−4
−2
2
4

2. Lattice approximation for KPZ

1D conservation law: $u = \frac{\partial}{\partial x} h$, $\frac{\partial}{\partial t} u = \frac{\partial}{\partial x} u^2 + \frac{1}{2}\frac{\partial^2}{\partial x^2} u + \frac{\partial}{\partial x} W$

- discretize $\mathbb{R}$ to $\mathbb{Z}$ coupled diffusions

$u(x,t) \rightsquigarrow u_j(t)$
- $W \rightsquigarrow db_j(t)$ independent Brownian motions
- $\frac{\partial^2}{\partial x^2} \rightsquigarrow$ difference Laplacian Δ
- $\frac{\partial}{\partial x} u^2$? local // invariant measure

$$du_j = \left(\lambda\left(u_{j+1}^2 + u_j u_{j+1} - u_{j-1} u_j - u_{j-1}^2\right) + (\Delta u)_j\right) dt + db_j - db_{j-1}$$

invariant measures : i.i.d. Gaussians

one-parameter family: mean

link to PASEP

generator L

invariant measure, expand in Hermite polynomials $\{n_j, j \in \mathbb{Z}\}$, $n_j \in \mathbb{N}$

P projects on subspace $\{n_j = 0, 1, j \in \mathbb{Z}\}$

PLP is generator of PASEP

variance bounds based on Landim, Quastel, Salmhofer, Yau (2004)

- stationary process, mean 0

$$t^{5/4} \le \sum_{j \in \mathbb{Z}} \mathbb{E}(u_0(0)\, u_j(t))\, j^2 \le t^{3/2}$$

|| should be $t^{4/3}$ ||

3. Lattice approximation for SHE

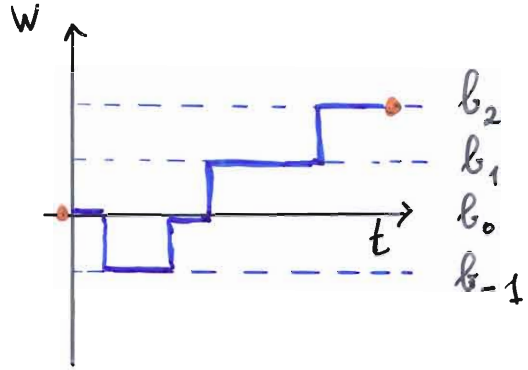
$Z(x,t) \rightsquigarrow Z_j(t)$ $\qquad dZ_j = (\Delta Z)_j\, dt + Z_j\, db_j \;,\; j \in \mathbb{Z}$

sharp wedge: $\qquad Z_j(0) = \delta_{j0}$

$j = 2$

symmetric random walk $w(t)$:

$$Z_j(t) = \mathbb{E}_0\left(e^{\int_0^t db_{w(s)}(s)}\, \delta_{w(t)j} \right)$$



point-to-point directed polymer

$$\rightsquigarrow \quad dh_j = \left(e^{h_{j+1} - h_j} + e^{h_{j-1} - h_j} - 2\right) dt + db_j$$

$$u_j = h_j - h_{j-1}$$

$$du_j = \left(e^{u_{j+1}} + e^{-u_j} - e^{u_j} - e^{-u_{j-1}}\right) dt + db_j - db_{j-1}$$

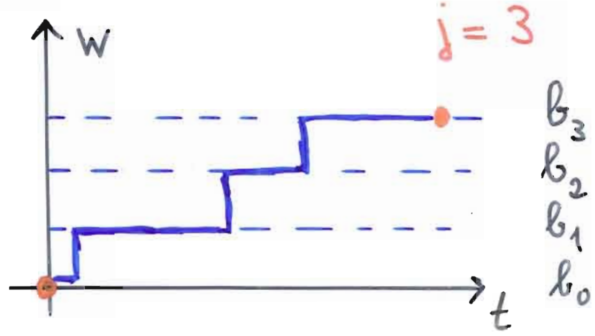
? invariant measure ?

⇒ asymmetric discretization

$$dZ_j = Z_{j-1}\,dt + Z_j\,db_j\,, \quad Z_j(0) = \delta_{j0}\,, \quad j \in \mathbb{Z}$$

asymmetric random walk $w(t)$

$$Z_j(t) = E_0\left(e^{\int_0^t db_{w(s)}(s)}\,\delta_{w(t),j}\right) e^t$$



|| zero temperature limit
|| maximize random energy

Baryshnikov (2001)
O'Connell, Yor (2001)

RMT

$$dh_j = e^{h_{j-1} - h_j}\,dt + db_j \qquad u_j = h_{j+1} - h_j$$

$$du_j = \left(e^{-u_j} - e^{-u_{j-1}}\right)dt + db_{j+1} - db_j$$

∃ stationary measures: product $\dfrac{1}{\Gamma(r)}\, e^{-e^{-x} - rx}\,dx \qquad \langle\cdot\rangle_r$

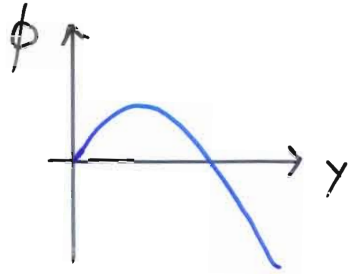
parameter $r > 0$

macroscopic shape

density $\quad \rho = \langle u_0 \rangle_T = -\psi(T)$

current $\quad j = - \langle e^{-u_0} \rangle_T = -T$

Digamma function $\quad \psi = \dfrac{\Gamma'}{\Gamma}$



shape function $\quad \phi(y) = \inf_{\rho} \left(-j(\rho) - \rho y \right), \; y \geq 0$

law of large numbers

$$\lim_{t \to \infty} \frac{1}{t} h_{\lfloor yt \rfloor}(t) = \phi(y) \qquad (*)$$

Theorem (O'Connell, Moriarty, 2007)

$$\lim_{n \to \infty} \frac{1}{n} \log h_n(\kappa n) = f(\kappa) \qquad \Bigg\| \qquad f(\kappa) = \inf_{s \geq 0} \left(\kappa s - \psi(s) \right)$$

in agreement with $(*)$ $\qquad y\kappa = 1$

fluctuations KPZ scaling theory Krug, Meakin, Halpin-Healy, 1992

$$\lim_{t\to\infty} \frac{1}{t^{1/3}}\left(h_{\lfloor yt \rfloor}(t) - t\,\phi(y)\right) = \left(\tfrac{1}{2}\lambda A^2\right)^{1/3} \underbrace{\xi_{TW}}_{\text{GUE Tracy-Widom r.v.}} \qquad (*)$$

- fix y, $\rho = \phi'(y)$
- $\lambda(\rho) = -j''(\rho)$
- $A = \langle u_0^2 \rangle_r - \langle u_0 \rangle_r^2$ $A(r) = \psi'(r)$

Theorem (Borodin, Corwin, 2011) ← Theorem 5.2.12 of Macdonald processes

$\kappa > \kappa^* > 0$

$$\lim_{n\to\infty} \frac{1}{n^{1/3}}\left(h_n(\kappa n) - n f(\kappa)\right) = \left(\frac{1}{2}\frac{1}{(-f''(\kappa))}\right)^{1/3} \xi_{TW}$$

in agreement with $(*)$

Conclusions

- KPZ limit of semi-discrete directed polymer

$\beta \to 0$, $n, t \to \infty$

$\beta^4 n = 1$

$\beta^4 t = 1$

- work directly with KPZ equation

ONLY replica

THANK YOU

for a really wonderful workshop!

happy birthday
to Funaki-sensei