Infinite Particle Systems associated with Airy kernel

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1. Introduction

Dyson's Brownian motion model [Dyson 62] is a one parameter family of the systems solving the following stochastic differential equation:

$$X_j(t) = x_j + B_j(t) + \frac{\beta}{2} \sum_{\substack{k:1 \leq k \leq n \\ k \neq j}} \int_0^t \frac{ds}{X_j(s) - X_k(s)}, \ 1 \leq j \leq n$$

where $B_j(t), j = 1, 2, \ldots, n$ are independent one dimensional Brownian motions.

In the system, interaction between any pair of particles is repulsive and its strength is proportional to the inverse of particle distance with proportional constant $\beta/2 > 0$.

We consider the case that $\beta = 2$ and call the model in the special case the Dyson model.

The Dyson model is realized by the following three processes:

(i) The process of eigenvalues of Hermitian matrix valued diffusion process in the Gaussian unitary ensemble (GUE).

(ii) The system of one-dimensional Brownian motions conditioned never to collide with each other.

(iii) The harmonic transform of the absorbing Brownian motion in a Weyle chamber of type A_{n-1}:

$$\mathbb{W}_n = \Big\{ \boldsymbol{x} = (x_1, x_2, \cdots, x_n) : x_1 < x_2 < \cdots < x_n \Big\}.$$

with harmonic function given by the Vandermonde determinant:

$$h_n(\boldsymbol{x}) = \prod_{1 \leq j < k \leq n} (x_k - x_j) = \det_{1 \leq j,k \leq n} \Big[x_k^{j-1} \Big].$$

The configuration space of unlabelled particles:

$$\mathfrak{M} = \left\{ \xi : \xi \text{ is a nonnegative integer valued Radon measures in } \mathbb{R} \right\}$$

Any element ξ of $\mathfrak{M}$ can be represented as:

$$\xi(\cdot) = \sum_{j \in \mathbb{I}} \delta_{x_j}(\cdot)$$

with some sequence $(x_j)_{j \in \mathbb{I}}$ of $\mathbb{R}$ satisfying $\sharp\{j \in \mathbb{I} : x_j \in K\} < \infty$, for any compact set K. The index set $\mathbb{I}$ is countable.

$\mathfrak{M}$ is a Polish space with the vague topology: we say ξ_n converges to ξ vaguely, if

$$\lim_{n \to \infty} \int_{\mathbb{R}} \varphi(x) \xi_n(dx) = \int_{\mathbb{R}} \varphi(x) \xi(dx)$$

for any $\varphi \in C_0(\mathbb{R})$, where $C_0(\mathbb{R})$ is the set of all continuous real-valued functions with compact supports.

For the solution $(X_j(t), j = 1, 2, , \dots, n)$ of

$$X_j(t) = x_j + B_j(t) + \sum_{\substack{k:1 \le k \le n \\ k \ne j}} \int_0^t \frac{ds}{X_j(s) - X_k(s)}, \ \ 1 \le j \le n,$$

we put

$$\xi^n(t) = \sum_{j=1}^{n} \delta_{X_j(t)}, \quad t \in [0, \infty),$$

which is an $\mathfrak{M}$-valued diffusion process starting from the configuration $\xi = \sum_{j=1}^{n} \delta_{x_j}$. We denote the process by $(\xi^n(t), \mathbb{P}_\xi)$ and call it the Dyson model with unlabeled particles.

The moment generating function of multitime distribution of a $\mathfrak{M}$-valued process $\xi(t)$ is defined as

$$\Psi^{\boldsymbol{t}}(\boldsymbol{f}) = \mathbb{E}\left[\exp\left\{\sum_{m=1}^{M}\int_{\mathbb{R}} f_m(x)\xi(t_m, dx)\right\}\right]$$

for $\boldsymbol{t} = (t_1, t_2, \dots, t_M)$ with $0 \le t_1 < t_2 < \cdots < t_M$, and $\boldsymbol{f} = (f_1, f_2, \dots, f_M)$ with $f_m \in \mathrm{C}_0(\mathbb{R}), 1 \le m \le M$.

Set $\chi_m(\cdot) = e^{f_m(\cdot)} - 1, 1 \le m \le M$.

$$\Psi^{\boldsymbol{t}}(\boldsymbol{f}) = \sum_{\substack{N_m \ge 0,\\ 1\le m\le M}} \int_{\prod_{m=1}^{M}\mathbb{R}^{N_m}} \prod_{m=1}^{M}\left\{\frac{1}{N_m!}d\boldsymbol{x}_{N_m}^{(m)}\prod_{i=1}^{N_m}\chi_m\left(x_i^{(m)}\right)\right\}$$
$$\times\rho\left(t_1, \boldsymbol{x}_{N_1}^{(1)}; \dots; t_M, \boldsymbol{x}_{N_M}^{(M)}\right),$$

with the multitime correlation functions $\rho\left(t_1, \boldsymbol{x}_{N_1}^{(1)}; \dots; t_M, \boldsymbol{x}_{N_M}^{(M)}\right)$.

A processs $\xi(t)$ is said to be determinantal if the moment generating function of the multitime distribution is given by a Fredholm determinant

$$\Psi^{\boldsymbol{t}}[\boldsymbol{f}] = \underset{\substack{(s,t)\in(t_1,t_2,\dots,t_M)^2,\\ (x,y)\in\mathbb{R}^2}}{\text{Det}} \Big[\delta_{st}\delta_x(y) + \mathbb{K}(s,x;t,y)\chi_t(y)\Big],$$

In other words, the multitime correlation functions are represented as

$$\rho\left(t_1, \boldsymbol{x}^{(1)}_{N_1}; \dots; t_M, \boldsymbol{x}^{(M)}_{N_M}\right) = \underset{\substack{1\le j\le N_m, 1\le k\le N_n\\ 1\le m,n\le M}}{\det} \left[\mathbb{K}(t_m, x^{(m)}_j; t_n, x^{(n)}_k)\right],$$

$0 < t_1 < \cdots < t_M < \infty$, $\boldsymbol{x}^{(m)}_{N_m} = (x^{(m)}_1, \dots, x^{(m)}_{N_m}) \in \mathbb{R}^{N_m}$, $1 \le m \le M$, $(N_1, \dots, N_M) \in \mathbb{N}^M$, $M \in \mathbb{N}$. The function $\mathbb{K}$ is called the correlation kernel of the determinantal process.

The Dyson model $\xi^n(t)$ starting from the origin is the determinantal process with the correlation kernel $\mathbb{K}_n$:

$$\mathbb{K}_n(s,x;t,y) = \begin{cases} \dfrac{1}{\sqrt{2s}} \displaystyle\sum_{k=0}^{n-1} \left(\dfrac{t}{s}\right)^{k/2} \varphi_k\left(\dfrac{x}{\sqrt{2s}}\right) \varphi_k\left(\dfrac{y}{\sqrt{2t}}\right), & \text{if } s \le t, \\ -\dfrac{1}{\sqrt{2s}} \displaystyle\sum_{k=n}^{\infty} \left(\dfrac{t}{s}\right)^{k/2} \varphi_k\left(\dfrac{x}{\sqrt{2s}}\right) \varphi_k\left(\dfrac{y}{\sqrt{2t}}\right), & \text{if } s > t. \end{cases}$$

where

$$\varphi_k(x) = \{\sqrt{\pi} 2^k k!\}^{-1/2} H_k(x) e^{-x^2/2}$$

is the normalized orthogonal functions on $\mathbb{R}$ comprising the Hermite polynomials $H_k(x)$

[Bulk scaling limit] As $n \to \infty$, the process $\xi^n(n+t)$ converges the infinite dimensional determinantal process $(\xi^{\sin}(t), P)$ whose correlation kernel $\mathcal{K}^{\sin}$ comprising trigonometrical functions:

$$\mathcal{K}^{\sin}(s,x;t,y) = \begin{cases} \dfrac{1}{\pi}\displaystyle\int_0^1 du\, e^{(t-s)u^2/2} \cos(u(x-y)), & \text{if } s<t, \\ \dfrac{\sin(x-y)}{\pi(x-y)}, & \text{if } s=t, \\ -\dfrac{1}{\pi}\displaystyle\int_1^\infty du\, e^{(t-s)u^2/2} \cos(u(x-y)), & \text{if } s>t. \end{cases}$$

The process is reversible process with reversible measure $\mu_{\sin}$ the determinanatal point process with the sine kernel

$$K^{\sin}(x,y) = \mathcal{K}^{\sin}(0,x;0,y).$$

[Nagao-Forrester (1998)]

Theorem [Katori-T:to appear in MPRF]
The process $(\xi^{\sin}(t), P)$ is a continuos reversible Markov process.

[Soft edge scaling limit] As $n \to \infty$, the scaled process

$$\theta_{a(n,t)}\xi^n(n^{1/3}+t) \equiv \{X_j(n^{1/3}+t) - a(n,t)\}_{j=1}^n,$$

with $a(n,t) = 2n^{2/3} + n^{1/3}t - t^2/4$, converges to the infinite dimensional determinantal process $(\xi^{\mathrm{Ai}}(t), P)$ whose correlation kernel $\mathcal{K}^{\mathrm{Ai}}$ comprising the Airy function $\mathrm{Ai}(x)$:

$$\mathcal{K}^{\mathrm{Ai}}(s,x;t,y) = \begin{cases} \int_{-\infty}^{0} du\, e^{(t-s)u/2}\mathrm{Ai}(x-u)\mathrm{Ai}(y-u), & \text{if } s \le t, \\ -\int_{0}^{\infty} du\, e^{(t-s)u/2}\mathrm{Ai}(x-u)\mathrm{Ai}(y-u), & \text{if } s > t. \end{cases}$$

The process is reversible process with reversible measure $\mu_{\sin}$ the determinanatal point process with the Airy kernel

$$K^{\mathrm{Ai}}(x,y) = \begin{cases} \dfrac{\mathrm{Ai}(x)\mathrm{Ai}'(y) - \mathrm{Ai}'(x)\mathrm{Ai}(y)}{x-y} & \text{if } x \ne y \\ (\mathrm{Ai}'(x))^2 - x(\mathrm{Ai}(x))^2 & \text{if } x = y, \end{cases}$$

[Forrester-Nagao-Honner (1999)],[Prähofer-Spohn (2002)]

Theorem [Katori-T:to appear in MPRF]

The process $(\xi^{\mathrm{Ai}}(t), P)$ is a continuos reversible Markov process.

2. Dirichlet forms

A function f defined on the configuration space $\mathfrak{M}$ is local if $f(\xi) = f(\xi_K)$ for some compact set K. A local function f is smooth if $f(\sum_{j=1}^{n} \delta_{x_j}) = \tilde{f}(x_1, x_2, \dots, x_n)$ with some smooth function $\tilde{f}$ on $\mathbb{R}^n$. We put

$$\mathcal{D}_0 = \{f : f \text{ is local and smooth with compact support}\}.$$

Put

$$\mathbb{D}[f,g](\xi) = \frac{1}{2} \sum_{j=1}^{\xi(K)} \frac{\partial \tilde{f}}{\partial x_j} \frac{\partial \tilde{g}}{\partial x_j}, \quad f, g \in \mathcal{D}_0,$$

and for a probability measure μ on $\mathfrak{M}$ we introduce the bilinear form

$$\mathcal{E}^{\mu}(f,g) = \int_{\mathfrak{M}} \mathbb{D}[f,g] d\mu, \quad f, g \in \mathcal{D}_0.$$

Let Φ be a free potential, Ψ be an interaction potential. For a given sequence $\{b_r\}$ of positive integers we introduce a Hamiltonian on $S_r = (-b_r, b_r)$:

$$H_r(\xi) = H_r^{\Phi,\Psi}(\xi) = \sum_{x_j \in S_r} \Phi(x_j) + \sum_{x_j, x_k \in S_r, j<k} \Psi(x_j, x_k)$$

We put $\mathfrak{M}_r^m = \{\xi \in \mathfrak{M} : \xi(S_r) = m\}$ and $\mu_r^m = \mu(\cdot \cap \mathfrak{M}_r^m)$.

Definition(quasi Gibbs measure) A probability measure μ is said to be a (Φ, Ψ)-quasi Gibbs measure if there exists an increasing sequence $\{b_r\}$ of positive integers and measures $\{\mu_{r,k}^m\}$ such that for each $r, m \in \mathbb{N}$ satisfying

$$\mu_{r,k}^m \le \mu_{r,k+1}^m, \quad k \in \mathbb{N}, \quad \lim_{k\to\infty} \mu_{r,k}^m = \mu_r^m, \textit{weekly}$$

and that for all $r, m, k \in \mathbb{N}$ and for $\mu_{r,k}^m$-a.s. $\xi \in \mathfrak{M}$

$$c^{-1} e^{-H_r(\zeta)} \mathbf{1}_{\mathfrak{M}_r^m}(\zeta)\Lambda(d\zeta) \le \mu_{r,k}^m(\pi_{S_r} \in d\zeta | \xi_{S_r^c}) \le c e^{-H_r(\zeta)} \mathbf{1}_{\mathfrak{M}_r^m}(\zeta)\Lambda(d\zeta)$$

Here Λ is the Poisson random measure with intensity measure dx.

(A.1) μ has a locally bounded correlation functions $\rho(\boldsymbol{x}_n), n \in \mathbb{N}$.

(A.2) μ is a (Φ, Ψ)-quasi Gibbs measure.

(A.3) There exist upper semicontinuous functions Φ_0, Ψ_0, and positive constants C and C' such that for any $x, y \in \mathbb{R}$

$$C^{-1}\Phi_0(x) \leq \Phi(x) \leq C\Phi_0(x),$$

$$C'^{-1}\Psi_0(x-y) \leq \Psi(x,y) \leq C'\Psi_0(x-y), \quad \Psi_0(x) = \Psi_0(-x)$$

Moreover, Φ_0 and Ψ_0 are locally bounded from below and $\{\Psi_0(x) = \infty\}$ is compact.

Theorem [Osada :arXiv:math.PR/0902.3561]
Assume (A.1), (A.2) and (A.3). Then

(1) $(\mathcal{E}^\mu, \mathcal{D}_0, L^2(\mathfrak{M}, \mu))$ is closable,

(2) its closure $(\mathcal{E}^\mu, \mathcal{D}^\mu, L^2(\mathfrak{M}, \mu))$ is a local quasi regular Dirichlet space,

(3) there exists a μ-reversible diffusion process $(\Xi(t), P)$ associated with the Diriclet space.

Corollary [Osada :arXiv:math.PR/0902.3561]
The probability measure μ_{sin} satisfies (A.1), (A.2) and (A.3) with $\Phi(x) = 0$ and $\Psi(x) = -2\log|x-y|$, and there exists a μ_{sin}-reversible diffusion process $(\Xi^{\text{sin}}(t), P)$ associated with the Diriclet space.

Theorem 1

The probability measure μ_{Ai} satisfies (A.1), (A.2) and (A.3) with $\Phi(x) = 0$ and $\Psi(x) = -2\log|x-y|$, and there exists a μ_{Ai}-reversible diffusion process $(\Xi^{\mathsf{Ai}}(t), P)$ associated with the Diriclet space.

Remark. It is proved that the process $(\xi^{\mathsf{Ai}}(t), P)$ is associated with a Diriclet space $(\mathcal{E}, \mathcal{D})$ which is a closed extension of the pre-Dirichlet space $(\mathcal{E}^{\mu_{\mathsf{Ai}}}, \mathcal{D}_0, L^2(\mathfrak{M}, \mu_{\mathsf{Ai}}))$. Then we see that $\mathcal{D}^{\mu_{\mathsf{Ai}}} \subset \mathcal{D}$. Our conjecture is the coinsidence of the above two Dirichlet spaces, i.e. $\mathcal{D}^{\mu_{\mathsf{Ai}}} = \mathcal{D}$.

3. SDE for the process $(\Xi^{\mathsf{Ai}}(t), P)$

Theorem 2

The process $\Xi^{\mathsf{Ai}}(t) = \sum_{j\in\mathbb{N}} \delta_{X_j(t)}$ satisfies the following SDE:

$$dX_j(t) = dB_j(t) \;+\; \lim_{L\to\infty} \left\{ \sum_{k\neq j:|X_k(t)|<L} \frac{1}{X_j(t)-X_k(t)} - \int_{|u|<L} \frac{\widehat{\rho}(u)}{-u} du \right\} dt, \qquad j \in \mathbb{N},$$

where $B_j(t)$, $j \in \mathbb{N}$ are independent Brownian motions and

$$\widehat{\rho}(u) = \frac{\sqrt{-u}}{\pi} \mathbf{1}(u < 0).$$

Remark In the above SDE we can replace the function $\widehat{\rho}(u)$ to a function $\overline{\rho}(u)$ satisfying the following conditions:

$$(1) \qquad \int_{\mathbb{R}} \frac{|\overline{\rho}(u) - \widehat{\rho}(u)|}{-u} dx < \infty,$$

$$(2) \qquad \lim_{L\to\infty} \int_{|u|<L} \frac{\overline{\rho}(u) - \widehat{\rho}(u)}{-u} dx = 0.$$

The density functions $\rho^{\mathsf{Ai}}(u) = K_{\mathsf{Ai}}(u,u)$ of μ^{Ai}, and $\rho_x^{\mathsf{Ai}}(u)$ of the palm measure μ_x^{Ai} satisfy the condition (1), however, it has not been shown if they satisfy the condition (2). Note that in the case that the density function $\rho^{\mathsf{Ai}}(u)$ is considered the integral implies Cauchy principal value.

Let μ^k be the Campbell measure of μ:

$$\mu^k(A \times B) = \int_A \mu_{\boldsymbol{x}_k}(B)\rho(\boldsymbol{x}_k)d\boldsymbol{x}_k, \quad A \in \mathcal{B}(\mathbb{R}^k), B \in \mathcal{B}(\mathfrak{M}).$$

We call $\mathbf{d}^\mu \in L^1_{loc}(\mathbb{R} \times \mathfrak{M}, \mu^1)$ the log derivative of μ if $\mathbf{d}^\mu$ satisfies

$$\int_{\mathbb{R}\times\mathfrak{M}} \mathbf{d}^\mu(x,\eta)f(x,\eta)d\mu^1 = -\int_{\mathbb{R}\times\mathfrak{M}} \nabla_x f(x,\eta)d\mu^1,$$

for any $f \in C_0^\infty(\mathbb{R}) \otimes \mathcal{D}_0$.

For $f, g \in C_0^\infty(\mathbb{R}^k) \otimes \mathcal{D}_0$

$$\nabla^k[f,g](\boldsymbol{x}_k,\eta) = \frac{1}{2}\sum_{j=1}^k \frac{\partial f(\boldsymbol{x}_k,\eta)}{\partial xj}\frac{\partial g(\boldsymbol{x}_k,\eta)}{\partial xj},$$

$$\mathbb{D}^k[f,g](\boldsymbol{x}_k,\eta) = \nabla^k[f,g](\boldsymbol{x}_k,\eta) + \mathbb{D}[f(\boldsymbol{x}_k,\cdot),g(\boldsymbol{x}_k,\cdot)](\eta),$$

Let $(\mathcal{E}^k, C_0^\infty(\mathbb{R}^k) \otimes \mathcal{D}_0)$ be the bilinear form defined by

$$\mathcal{E}^k(f,g) = \int_{\mathbb{R}^k \times \mathfrak{M}} \mathbb{D}^k[f,g] d\mu^k.$$

(a.1) ρ^k is locally bounded for each $k \in \mathbb{N}$.

(a.2) $(\mathcal{E}^k, C_0^\infty(\mathbb{R}^k) \otimes \mathcal{D}_0)$ is closable on $L^2(\mu^k)$ for each $k \in \mathbb{N}$.

Theorem [Osada, JMSJ 2010]
Assume (a.1) and (a.2). Then the closure $(\mathcal{E}^k, \mathcal{D}^k, L^2(\mathbb{R}^k \times \mathfrak{M}, \mu^k)$ of $(\mathcal{E}^k, C_0^\infty(\mathbb{R}^k) \otimes \mathcal{D}_0, L^2(\mathbb{R}^k \times \mathfrak{M}, \mu^k))$ is a local quasi regular Dirichlet space, and the associated diffusion process $((\boldsymbol{X}^k(t), \widehat{\Xi}(t)), P)$ exists.

(a.3) There exists a log derivative $\mathbf{d}^{\mu}$.

(a.4) $$Cap(\mathfrak{M} \setminus \mathfrak{M}^{\infty}) = 0,$$

where $\mathfrak{M}^{\infty} = \{\xi \in \mathfrak{M} : \xi(x) \leq 1, \forall x \in \mathbb{R}, \xi(\mathbb{R}) = \infty\}$.

(a.5) There exists $T > 0$ such that for each $R > 0$

$$\liminf_{r \to \infty} \left(\int_{|x| \leq R+r} \rho(x) dx \int_{r/\sqrt{(r+R)T}}^{\infty} e^{-u^2/2} du \right) = 0$$

Theorem [Osada, PTRF (on line first)]
Assume (a.1) - (a.5). There exists $\mathfrak{M}_0 \subset \mathfrak{M}^{\infty}$ such that $\mu(\mathfrak{M}_0) = 1$, and for any $\xi = \sum_{j \in \mathbb{N}} \delta_{x_j} \in \mathfrak{M}_0$, there exists $\mathbb{R}^{\mathbb{N}}$-valued continuous process $\boldsymbol{X}(t)$ satisfying $\mathbf{X}(0) = \boldsymbol{x} = (x_j)_{j=1}^{\infty}$

$$dX_j(t) = dB_j(t) + \mathbf{d}^{\mu} \left(X_j(t), \sum_{k:k \neq j} \delta_{X_k(t)} \right) dt, \quad j \in \mathbb{N}$$

The key part in the proof of Th 2 is to determine the log derivative of μ_{Ai}.

Lemma 3 For $x \in \mathbb{R}$ and $\eta = \sum_{j\in\mathbb{N}} \delta_{y_j}$ with $\eta(\{x\}) = 0$,

$$\mathbf{d}^{\mu_{\mathsf{Ai}}}(x,\eta) = \lim_{L\to\infty} \left\{ \sum_{j:|x-y_j|\le L} \frac{1}{x-y_j} - \int_{|u|\le L} \frac{\widehat{\rho}(u)}{-u} du \right\}$$

Remark [Osada, PTRF (on line first)] For $x \in \mathbb{R}$ and $\eta = \sum_{j\in\mathbb{N}} \delta_{y_j}$ with $\eta(\{x\}) = 0$,

$$\mathbf{d}^{\mu_{\sin}}(x,\eta) = \lim_{L\to\infty} \left\{ \sum_{j:|x-y_j|\le L} \frac{1}{x-y_j} \right\}$$

To prove Lemma 3 we use the distribution of n particles in GUE system is

$$\mu^n_{GUE}(u_1, u_2, \ldots, u_n) = \frac{1}{Z} \prod_{i<j} |u_i - u_j|^2 \,\text{exp} \left\{ -\sum_{i=1}^{n} \frac{|u_i|^2}{2} \right\},$$

We put $u_j = 2\sqrt{n} + \frac{x_j}{n^{1/6}}$ and intrduce the measure defined by

$$\mu^n_{\mathcal{A}}(x_1, x_2, \ldots, x_n) = \frac{1}{Z} \prod_{i<j} |x_i - x_j|^2 \,\text{exp} \left\{ -\sum_{i=1}^{n} \frac{|2\sqrt{n} + n^{-1/6} x_i|^2}{2} \right\},$$

which is the determinantal point process with the correlation kernel

$$K^n_{\mathcal{A}}(x, y) = n^{1/3} \frac{\Psi_n(x)\Psi_{n-1}(y) - \Psi_{n-1}(x)\Psi_n(y)}{x - y}$$

with

$$\Psi_n(x) = n^{1/12} \varphi_n\Big(\sqrt{2n} + \frac{x}{\sqrt{2} n^{1/6}}\Big)$$

The palm measure $\mu^n_{\mathcal{A},z}$ is also a determinantal point process and its kernel is represented as

$$K^n_{\mathcal{A},z} = K^n_{\mathcal{A}}(x,y) - \frac{K^n_{\mathcal{A}}(x,z)K^n_{\mathcal{A}}(z,y)}{K^n_{\mathcal{A}}(x,z)}.$$

Note that

$$\lim_{n\to\infty} K^n_{\mathcal{A}}(x,y) = \mathcal{K}^{\mathsf{Ai}}(x,y) \quad \text{and} \quad \lim_{n\to\infty} K^n_{\mathcal{A},z}(x,y) = \mathcal{K}^{\mathsf{Ai}}_z(x,y)$$

and

$$\lim_{n\to\infty} \mu^n_{\mathcal{A}} = \mu^{\mathsf{Ai}} \quad \text{and} \quad \lim_{n\to\infty} \mu^n_{\mathcal{A},z} = \mu^{\mathsf{Ai}}_z.$$

In particular

$$\lim_{n\to\infty} \rho^n_{\mathcal{A}}(x) = \rho^{\mathsf{Ai}}(x) \quad \text{and} \quad \lim_{n\to\infty} \rho^n_{\mathcal{A},z}(x) = \rho^{\mathsf{Ai}}_z(x),$$

and

$$\mu^{n,1}(dxd\eta) \equiv \mu^n_{\mathcal{A},x}(d\eta)\rho_{\mathcal{A}}(x)dx \to \mu^{n,1}(dxd\eta), \quad \text{vaguely} \quad n \to \infty.$$

The log derivative $\mathbf{d}^n$ of the measure $\mu_{\mathcal{A}}^n$ is given by

$$\mathbf{d}^n(x,\eta) = \mathbf{d}^n\left(x, \sum_{j=1}^{n-1} \delta_{y_j}\right) = \sum_{j=1}^{n-1} \frac{1}{x-y_j} - n^{1/3} - \frac{n^{-1/3}}{2} x.$$

We divide $\mathbf{d}^n$ into three parts:

$$\mathbf{d}^n(x,\eta) = \mathrm{g}_L^n(x,\eta) + w_L^n(x,\eta) + u^n(x),$$

with

$$\mathrm{g}_L^n(x,\eta) = \sum_{|x-y_j|<L} \frac{1}{x-y_j} - \int_{|x-u|<L} \frac{\rho_{\mathcal{A},x}^n(u)}{x-u} du,$$

$$w_L^n(x,\eta) = \sum_{|x-y_j|\geq L} \frac{1}{x-y_j} - \int_{|x-u|\geq L} \frac{\rho_{\mathcal{A},x}^n(u)}{x-u} du,$$

$$u^n(x) = \int_{\mathbb{R}} \frac{\rho_{\mathcal{A},x}^n(u)}{x-u} du - n^{1/3} - \frac{n^{-1/3}}{2} x.$$

Lemma 3 is derived from the fact that

$$\mathbf{d}^{\mu^{\mathsf{Ai}}}(x,\eta) = \lim_{n\to\infty} \mathbf{d}^{n}(x,\eta) = \lim_{L\to\infty} \left\{ \sum_{|x-y_j|<L} \frac{1}{x-y_j} - \int_{|u|\le L} \frac{\widehat{\rho}(y)}{-y} du \right\}$$

if the following conditions hold:

$$\lim_{n\to\infty} \mathrm{g}_L^n(x,\eta) = \mathrm{g}_L(x,\eta), \qquad \text{in } L^{\widehat{p}}(\mu^1) \text{ for any } L>0, \tag{1}$$

$$\lim_{L\to\infty} \limsup_{n\to\infty} \int_{[-r,r]\times\mathfrak{M}} |w_L^n(x,\mathsf{y})|^{\widehat{p}} d\mu^{n,1}(dxd\eta) = 0, \tag{2}$$

$$\lim_{n\to\infty} u^n(x) = u(x), \qquad \text{in } L^{\widehat{p}}_{loc}(\mathbb{R},dx)\ , \tag{3}$$

with

$$\mathrm{g}_L(x,\eta) = \sum_{|x-y_j|<L} \frac{1}{x-y_j} - \int_{|x-u|<L} \frac{\rho_x^{\mathsf{Ai}}(u)}{x-u} du,$$

and

$$u(x) = \lim_{L\to\infty} \left\{ \int_{|u|\le L} \frac{\rho_x^{\mathsf{Ai}}(u)}{x-u} du - \int_{|u|\le L} \frac{\widehat{\rho}(u)}{-u} du \right\} \in L^{\widehat{p}}_{loc}(\mathbb{R},dx).$$

The first condition:

$$\lim_{n\to\infty} \mathrm{g}_L^n(x,\eta) = \mathrm{g}_L(x,\eta), \qquad \text{in } L^{\widehat{p}}(\mu^1) \text{ for any } L>0,$$

$(\because)$ Since

$$\mathrm{g}_L(x,\eta) - \mathrm{g}_L^n(x,\eta) = -\int_{|x-u|<L} \frac{\rho_x^{\mathsf{Ai}}(u) - \rho_{\mathcal{A},x}^n(u)}{x-u} du,$$

The claim is derived from

$$\lim_{n\to\infty} \rho_{\mathcal{A},x}^n(u) = \rho_x^{\mathsf{Ai}}(u)$$

and the behavior of $\rho_{\mathcal{A},x}^n(u)$ and $\rho_x^{\mathsf{Ai}}(u)$ around x.

The second condition:

$$\lim_{L\to\infty}\limsup_{n\to\infty}\int_{[-r,r]\times\mathfrak{M}}|w_L^n(x,\mathsf{y})|^{\hat{p}}d\mu^{n,1}(dxd\eta)=0,$$

$(\because)$

Since $\displaystyle\int_{\mathfrak{M}}\sum_{|x-y_j|\ge L}\frac{1}{x-y_j}d\mu^n_{\mathcal{A},x}(d\eta)=\int_{|x-u|\ge L}\frac{\rho^n_{\mathcal{A},x}(u)}{x-u}du$, we have

$$w_L^n(x,\eta)=\sum_{|x-y_j|\ge L}\frac{1}{x-y_j}-\int_{\mathfrak{M}}\sum_{|x-y_j|\ge L}\frac{1}{x-y_j}d\mu^n_{\mathcal{A},x}(d\eta)$$

Since $\mu^n_{\mathcal{A},x}(d\eta)$ is a determinantal point process, for any bounded closed interval D of $\mathbb{R}$, we have

$$\int_{\mathfrak{M}}\mu^n_{\mathcal{A},x}(d\eta)\left|\eta(D)-\int_D\rho^n_{\mathcal{A},x}(u)du\right|^{2k}\le\left(3\int_D\rho^n_{\mathcal{A},z}(u)du\right)^k\le\left(3\int_D\rho^{\mathsf{Ai}}_x(u)du\right)^k,$$

for $k,n\in\mathbb{N}$.

Let $\xi \in \mathfrak{M}$ and ρ be the nonnegative function on $\mathbb{R}$. Suppose that there exist $\varepsilon \in (0,1)$, $C_1 > 0$ and $L_1 > 0$ such that

$$\left|\xi([0,L]) - \int_0^L \rho(x)dx\right| \le C_1 L^{\varepsilon}, \quad \left|\xi([-L,0)) - \int_{-L}^0 \rho(x)dx\right| \le C_1 L^{\varepsilon}, \quad L \ge L_1.$$

then ξ satisfies

$$\left|\int_{|x|\ge L} \frac{\rho(x)dx - \xi(dx)}{x}\right| \le \frac{3C_1}{1-\varepsilon} L^{\varepsilon-1}.$$

The third condition:

$$\lim_{n\to\infty} u^n(x) = u(x), \qquad \text{in } L^{\widehat{p}}_{loc}(\mathbb{R}, dx) \ ,$$

$(\because)$ We put

$$\widehat{\rho}^n_{\mathsf{SC}}(x) = \frac{1}{\pi}\sqrt{-x\left(1+\frac{x}{4n^{2/3}}\right)}\mathbf{1}(-4n^{2/3} < x < 0).$$

We note that $\widehat{\rho}^n_{\mathsf{SC}}(x) \nearrow \widehat{\rho}(x)$, $n \to \infty$ and

$$\int_{\mathbb{R}} dx \ \widehat{\rho}^n_{\mathsf{SC}}(x) = n, \quad \text{and} \quad \int_{\mathbb{R}} \frac{\widehat{\rho}^n_{\mathsf{SC}}(u)}{-u} du = n^{1/3}.$$

Then

$$u^n(x) = \int_{\mathbb{R}} \frac{\rho^n_{\mathcal{A},x}(u)}{x-u} du - \int_{\mathbb{R}} \frac{\widehat{\rho}^n_{\mathsf{SC}}(u)}{-u} du - \frac{n^{-1/3}}{2} x.$$

$$\to \lim_{L\to\infty} \left\{ \int_{|u|\le L} \frac{\rho^{\mathsf{Ai}}_x(u)}{x-u} du - \int_{|u|\le L} \frac{\widehat{\rho}(u)}{-u} du \right\} = u(x) \quad n \to \infty.$$

Remarak. Consider the diffusion process associated with the Dirichlet space

$$\mathcal{E}^{\mu^n_{\mathcal{A}}}(f,g) = \int_{\mathfrak{M}} D[f,g] d\mu^n_{\mathcal{A}}.$$

The infinitesimal generator associated with the process is geven by

$$L_n = \frac{1}{2}\sum_{i=1}^n \frac{d^2}{dx_i^2} + \sum_{i=1}^n \mathbf{d}^n(x_i, \{x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n\}) \frac{d}{dx_i}$$

and the process is associated with $(Y_j(t) - n^{1/3})_{j=1}^n$, where $\mathbf{Y}(t)$ is another Dyson model, noncolliding Ornstein-Uhlenbeck processes:

$$dY_j(t) = dB_j(t) + \sum_{\substack{k:1\le k\le n \\ k\neq j}} \frac{dt}{Y_j(t) - Y_k(t)} - \frac{n^{-1/3}}{2} dY_j(t), \; 1 \le j \le n$$

We can show that

$$\sum_{j=1}^n \delta_{Y_j(t)-n^{1/3}} \to \xi^{\mathrm{Ai}}(t), \quad n \to \infty.$$

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