1次元KPZ方程式とその普遍性

T. Sasamoto

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概 要

独立同分布に従う N 個の確率変数の和の揺らぎは、N が大きい極限で、よく知られているように中心極限定理によってオーダーは $O(N^{1/2})$ であり、分布は正規分布で与えられる。これに対し、1 次元の界面成長を記述する KPZ 方程式の時刻 t における界面高さ揺らぎは、t が大きい極限で、オーダーは $t^{1/3}$ であり、分布はランダム行列理論に現れる GUE Tracy-Widom 分布で与えられる。本講義では、後者の揺らぎの性質について、その背景の 説明から始めて実際に極限分布を導出する所までを解説する。

目 次

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1 Introduction

In probability we are often interested in asymptotic behaviors of certain random variables. An important example is the sum of independent random variables for which the asymptotic law is Gaussian. A natural question is what happens when the random variables are not independent.

Many examples are provided in the studies of interacting particle systems in which many elements of a system have some "interaction" among them. These systems are of interest for themselves. At the same time they have a lot of applications to physics, chemistry and other branches of science.

In this note we study fluctuation properties of a certain type of interacting particle systems. One of our main interests goes to the asymmetric simple exclusion process (ASEP) in which many particles do asymmetric random walk with exclusion interaction. Naturally ASEP can be considered as a model of transport. But through a simple mapping it can also be considered as a model of surface growth.

When we consider large time and long distance behaviors of the systems, universal features appear. This universality is referred to as the KPZ universality because Kardar, Parisi and Zhang proposed a famous nonlinear stochastic partial differential equation (KPZ equation) to describe surface growth phenomena. An intriguing feature of this is that universal quantities can be studied very explicitly due to the connection to random matrix theory and integrable systems. In addition an explicit solution for the KPZ equation was found very recently.

In this note we study some basic parts of these developments.

2 Random walk

First let us start from recalling some basic facts about the case where there is only one particle. Here we consider a few examples of the random walk and the Brownian motion. This should give us a solid ground for proceeding to many particle case. Let $(\Omega, \mathcal{F}, P)$ be a probability space and E the corresponding expectation. For a Markov chain $X(t)$, the transition probability is defined by

$$G(x; t|y; 0) = \mathbb{P}[X(t) = x | X(0) = y]. \tag{2.1}$$

(A remark: In the theory of stochastic processes it is more common to put x and y in the opposite order. But here we use the above notation which is often used for ASEP.)

Random walk. First we consider a random walk. In its standard version, a random walker, which we also call a particle, hops to the left and the right with probability $1/2$. Here we introduce a slightly modified version. It starts from the site $y \in \mathbb{Z}$ and at each time step it moves to the right neighboring site with probability p and stays put with probability $1 - p$ where $0 \leq p \leq 1$. More explicitly this is defined as follows. Suppose

$t \in \mathbb{N} = \{0, 1, 2, \ldots\}$ which represents time. Let $Y_n, n \in \mathbb{Z}_+ = \{1, 2, \ldots\}$be i.i.d. random variables, each of which is Bernoulli distributed with parameter p,

$$\mathbb{P}[Y_n = 1] = p, \mathbb{P}[Y_n = 0] = 1 - p. \tag{2.2}$$

Then

$$X(0) = y \in \mathbb{Z}, X(t) = y + Y_1 + \ldots + Y_t, \ t \geq 1 \tag{2.3}$$

is the random walk here.

The transition probability of this random walk can be found by a simple combinatorial consideration, i.e., by considering all possibilities that the walker starts from y and arrives at x at time t. It is given by

$$G(x; t|y; 0) = G(x - y; t|y = 0, 0) = \binom{t}{x - y} p^{x-y}(1 - p)^{t-x+y} \tag{2.4}$$

where $\binom{t}{x} = \frac{t!}{x!(t-x)!}$ is the binomial coefficient. Since Y_n are i.i.d., one easily finds

$$E[X(t)] = tp, \quad V[X(t)] := E[(X(t) - E[X(t)])^2] = tp(1 - p). \tag{2.5}$$

Let us set

$$W_x = \inf\{t|X(t) = x\} - \inf\{t|X(t) = x - 1\} - 1 \tag{2.6}$$

for $x \geq 1$, which represents the waiting time of the particle to make a hop from site $x-1$ to x. Setting $\tau_0 = 0, \tau_x = W_1 + \ldots + W_x, x \geq 1$, one sees that $X(t)$ can be written as

$$X(t) = y + \inf\{x \in \mathbb{N}; \tau_{x+1} + x + 1 > t\}. \tag{2.7}$$

In many cases we are interested in the long time asymptotics. For the case of the random walk, by the central limit theorem, one has

$$\lim_{t\to\infty} \mathbb{P}\left[\frac{X(t) - tp}{\sqrt{tp(1 - p)}} \leq s\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{s} e^{-x^2/2} dx. \tag{2.8}$$

Brownian motion. Our second example is the Brownian motion(BM). It is a stochastic process $(B(t), t \geq 0)$ and is defined by the following two conditions. (B1) It is the Gaussian stochastic process with

$$E[B(t)] = 0, \tag{2.9}$$

$$\text{Cov}[B(s), B(t)] := E\left[(B(s) - E[B(s)])(B(t) - E[B(t)])\right] = \min\{s, t\}. \tag{2.10}$$

(B2) The path is continuous in t almost surely. The first condition is equivalent to requiring that it has a stationary independent increments and the distribution of $B(t)$ is $N(0, t)$, i.e., the Gaussian with mean 0 and variance t. One can also define the BM starting from an arbitrary position $y \in \mathbb{R}$ by

$$B_y(t) = y + B(t). \tag{2.11}$$

From the above definition, it is obvious that the transition density $G(x;t|y;0)$ of $B_y(t)$ is given by

$$G(x;t|y;0)dx = \mathbb{P}[y + B(t) \in dx] = \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-y)^2}{2t}} dx. \tag{2.12}$$

Note that it satisfies

$$\frac{\partial}{\partial t} G(x;t) = \frac{1}{2}\frac{\partial^2}{\partial x^2} G(x;t), \quad G(x;t=0) = \delta(x-y) \tag{2.13}$$

where $\delta(x)$ is the Dirac's delta function. In fact the Brownian motion can be considered as a certain limit of the random walk (Donsker's theorem). We call this the diffusion limit.

Poisson random walk. Let $W_x, x \in \mathbb{N}$ be i.i.d. random variables, each of which is exponentially distributed with mean one,

$$\mathbb{P}[W_x \in dt] = e^{-t} 1_{[0,\infty)}(t) dt. \tag{2.14}$$

Set $\tau_0 = 0, \tau_x = W_1 + \ldots + W_x, x \geq 1$. For $y \in \mathbb{Z}, t \in [0,\infty)$, define

$$X(t) = y + \inf\{x \in \mathbb{N}; \tau_{x+1} > t\} \tag{2.15}$$

Then $G(x;t|y;0) = \mathbb{P}[X(t) = x | X(0) = y]$ satisfies

$$\frac{d}{dt} G(x;t|y;0) = G(x-1;t|y;0) - G(x;t|y;0), \quad G(x;0|y;0) = \delta_{xy}. \tag{2.16}$$

and hence

$$G(x,t|y,0) = \frac{t^{x-y}}{(x-y)!} e^{-t}. \tag{2.17}$$

Notice that the transition probability can be obtained by solving this equation.

It is often easier to written down the forward or backward equation. Finding an explicit formula for the transition probability is usually a nontrivial task.

Continuous time asymmetric random walk. When the particle hops to the right with rate p and to the left with rate q $(p,q \geq 0, p+q=1)$, the transition probability $G(x;t|y,0)$ satisfies

$$\frac{d}{dt} G(x;t|y;0) = pG(x-1;t|y;0) + qG(x+1;t|y;0) - G(x;t|y;0) \tag{2.18}$$

Set

$$\epsilon(\xi) = p/\xi + q\xi - 1. \tag{2.19}$$

Proposition 2.1.

$$G(x;t|y,0) = \frac{1}{2\pi i} \int_C \xi^{x-y-1} e^{\epsilon(\xi)t} d\xi \tag{2.20}$$

where C is a contour enclosing the origin anticlockwise.

This can be checked by a simple computation. This can also be written as

$$G(x;t|y;0) = e^{-(p+q)t} \left(\frac{q}{p}\right)^{-\frac{x-y}{2}} I_{x-y}\left(2\sqrt{\frac{q}{p}}\right), \tag{2.21}$$

where $I_n(x)$ is the modified Bessel function.

3 Formulation of lattice gases

In this section we explain the formulation of the lattice gas on $\mathbb{Z}$. We do not give the details but refer the readers to existing literature [1–5].

Let us introduce a notation,

$$\eta(x) = \begin{cases} 0, & \text{site } x(\in \mathbb{Z}) \text{ is empty,} \\ 1, & \text{site } x(\in \mathbb{Z}) \text{ is occupied.} \end{cases} \tag{3.1}$$

Our state space is $X = \{0,1\}^{\mathbb{Z}}$, which is compact in the product topology. $\eta(\in X)$ is called a configuration (of particles). We set

$$\eta^{xy}(u) = \begin{cases} \eta(y), & \text{if } \ u = x, \\ \eta(x), & \text{if } \ u = y, \\ \eta(u), & \text{if } \ u \neq x, y. \end{cases} \tag{3.2}$$

Let us set $C(X)$ to be the space of continuous functions on X. Let us consider a Feller process η_t: $[0,\infty) \to X$ for which $E^{\eta} f(\eta_t) \in C(X)$ for any $f \in C(X)$. Let P^{μ} be the distribution of the process with initial measure μ and E^{μ} be the corresponding expectation. We sometimes consider the situation in which the process starts from a single configuration η for which case we abuse the notation like P^{η} and E^{η}. For η_t, one can define the semigroup $S(t)$ on $C(X)$ by

$$S(t)f(\eta) = E^{\eta} f(\eta_t), \quad f \in C(X). \tag{3.3}$$

It is known that there is a one-to-one correspondence between the semigroup $S(t)$ and the generator L defined by

$$Lf = \lim_{t \to 0} \frac{S(t)f - f}{t}. \tag{3.4}$$

The lattice gas can be constructed by giving its generator. Let $c(x, y, \eta)$ be the rate of exchange of the occupancies at x and y. The generator of the process is given by introducing an operator

$$Lf(\eta) = \frac{1}{2} \sum_{x,y \in \mathbb{Z}} c(x, y, \eta)[f(\eta^{xy}) - f(\eta)] \tag{3.5}$$

for f a cylinder function (which depends only on finitely many coordinates) and then taking its closure. For this construction to work the rate $c(x, y\eta)$ should satisfy certain conditions but here we simply assume that they are satisfied.

In the $t \to \infty$ limit, the system approaches the stationary measure μ, which is defined by

$$E^{\mu}[Lf] = 0 \tag{3.6}$$

For the ASEP the operator L on the cylinder set is given by

$$Lf(\eta) = \frac{1}{2} \sum_{x \in \mathbb{Z}} \left(p\eta(x)(1 - \eta(x+1)) + q(1 - \eta(x))\eta(x+1) \right) [f(\eta^{xy}) - f(\eta)]. \tag{3.7}$$

Here $p, q \geq 0, p+q=1$. The special case of the ASEP in which either $p=0$ or $q=0$ is called TASEP (Totally ASEP).

For ASEP on $\mathbb{Z}$, there are two series of the stationary measures. One is the product measure with $E[\eta(x)] = 1/(1+(q/p)^x)$ and its translations. The other is the Bernoulli measure with density $\rho, 0 \leq \rho \leq 1$. It is known that these exhaust the all extremal stationary measures.

4 Transition probability

In this section we consider the transition probability of the ASEP on $\mathbb{Z}$, which is the probability that N particles starting from $y_1, \ldots, y_N$ at time 0 are on sites $x_1, \ldots, x_N$ at time t. (We assume $x_i < x_{i+1}, y_i < y_{i+1}, 1 \leq i \leq N-1$.) Let $X_i(t), 1 \leq i \leq N$ be the position of the i th particle at time t. Then the transition probability is

$$G(x_1, \ldots, x_N; t | y_1, \ldots, y_N; 0) = \mathbb{P}[X_i(t) = x_i, 1 \leq i \leq N | X_i(0) = y_i, 1 \leq i \leq N]. \quad (4.1)$$

Note that, for the ASEP, we can consider the case where there are N particles since the number of particles is conserved. We often omit the dependence on y as well.

For stochastic interacting particle systems, it is in general difficult to obtain an expression for the transition probability. But for the ASEP one can find useful expressions. As we will see shortly, for the TASEP on $\mathbb{Z}$, the transition probability can be written as a single determinant. For ASEP with general p and q, the formula using the Bethe ansatz contains a summation over permutations and is more involved than for TASEP. But we can still utilize it to study current fluctuations.

4.1 Forward equation

First let us write down the forward equation satisfied by the transition probability. The equation for one particle ($N=1$) case reads

$$\frac{d}{dt} G(x;t) = pG(x-1;t) + qG(x+1;t) - G(x;t). \quad (4.2)$$

This is the same as (2.18) and the solution is given by (2.20). Next for $N=2$, we have to consider two cases separately. When $x_2 - x_1 \geq 2$, the forward equation reads

$$\begin{aligned} \frac{d}{dt} G(x_1, x_2; t) &= pG(x_1-1, x_2; t) + qG(x_1+1, x_2; t) + pG(x_1, x_2-1; t) \\ &\quad + qG(x_1, x_2+1; t) - 2G(x_1, x_2; t). \end{aligned} \quad (4.3)$$

When $x_2 = x_1 + 1$, due to the exclusion rule, the equation is

$$\frac{d}{dt} G(x_1, x_1+1; t) = pG(x_1-1, x_1+1; t) + qG(x_1, x_1+2; t) - G(x_1, x_2; t). \quad (4.4)$$

The initial condition for the transition probability is

$$G(x_1, x_2; t|y_1, y_2; 0) = \delta_{x_1 y_1} \delta_{x_2 y_2}. \tag{4.5}$$

The transition probability is determined as the solution to (4.3),(4.4),(4.5). It is a little cumbersome that one has to deal with the two equations (4.3),(4.4) separately. But the second one can be replaced by a boundary condition for $G(x_1, x_2; t)$. Setting $x_2 = x_1 + 1$ in (4.3) one gets

$$\begin{aligned} \frac{d}{dt} G(x_1, x_1+1; t) &= pG(x_1-1, x_1+1; t) + qG(x_1+1, x_1+1; t) + pG(x_1, x_1; t) \\ &\quad + qG(x_1, x_1+2; t) - 2G(x_1, x_1+1; t). \end{aligned} \tag{4.6}$$

Comparing (4.6) with (4.4), we have

$$pG(x_1, x_1, t) + qG(x_1+1, x_1+1; t) = G(x_1, x_1+1; t). \tag{4.7}$$

This means that instead of considering (4.3),(4.4) for $x_1 < x_2$, one can consider (4.3) with the boundary condition (4.7) for $x_1 \le x_2$ and focus on the case $x_1 < x_2$.

For general N the situation is similar to the $N = 2$ case. The main forward equation reads

$$\begin{aligned} \frac{d}{dt} G(x_1, \ldots, x_N; t) = \sum_{i=1}^{N} \Big(&pG(\ldots, x_i - 1, \ldots; t) + qG(\ldots, x_i + 1, \ldots, t) \\ &- G(\ldots, x_i, \ldots; t) \Big). \end{aligned} \tag{4.8}$$

One has to solve this with the boundary condition

$$pG(\ldots, x_i, x_i, \ldots; t) + qG(\ldots, x_i + 1, x_i + 1, \ldots; t) = G(\ldots, x_i, x_{i+1}, \ldots; t) \tag{4.9}$$

and the initial condition,

$$G(x_1, \ldots, x_N; t = 0) = \prod_{i=1}^{N} \delta_{x_i y_i}. \tag{4.10}$$

4.2 Schütz formula for TASEP [6]

For TASEP, Schütz found the following formula for the transition probability [6].

Proposition 4.1.

$$G(x_1, \ldots, x_N; t|y_1, \ldots, y_N; 0) = \det \left(F_{l-j}(x_l - y_j; t) \right)_{1 \le j, l \le N} \tag{4.11}$$

Here the function $F_n(x, t)$ appearing as a matrix element of the determinant is

$$F_n(x, t) = e^{-t} \sum_{k=0}^{\infty} \binom{k+n-1}{k} \frac{t^{k+x}}{(k+x)!} \tag{4.12}$$

To prove the formula it is useful to list a few properties of $F_n(x,t)$.

Lemma 4.2. *(i) The function $F_n(x,t)$ can be written as a contour integral,*

$$F_n(x,t) = \frac{1}{2\pi i}\int_{0,1} dz \frac{1}{z^{x+1}}\,(1-1/z)^{-n}\,e^{-(1-z)t}$$

where the contour enclosing the poles at $z=0,1$ of the integrand anticlockwise
(ii)

$$F_{n+1}(x,t) = \sum_{y=x}^{\infty} F_n(y,t) \tag{4.13}$$

(iii)

$$\frac{d}{dt}F_n(x,t) = F_n(x-1,t) - F_n(x,t) \tag{4.14}$$

Proof.
(i)

$$\begin{aligned}\left(1-\frac{1}{z}\right)^{-n} &= \sum_{k=0}^{\infty}\binom{-n}{k}(-1/z)^k = \sum_{k=0}^{\infty}\frac{(-n)(-n-1)\cdots(-n-k+1)}{k!}\frac{(-1)^k}{z^k}\\ &= \sum_{k=0}^{\infty}\binom{k+n-1}{k}\frac{1}{z^k}.\end{aligned} \tag{4.15}$$

Hence the LHS is

$$\begin{aligned}&e^{-t}\sum_{k=0}^{\infty}\binom{k+n-1}{k}\frac{1}{2\pi i}\int_{0,1} dz\frac{e^{zt}}{z^{x+k+1}}\\ &= e^{-t}\sum_{k=0}^{\infty}\binom{k+n-1}{k}\frac{t^{k+x}}{(k+x)!}\end{aligned} \tag{4.16}$$

which is RHS.
(ii)

$$\begin{aligned}\text{RHS} &= \sum_{y=x}^{\infty}\frac{1}{2\pi i}dz\frac{1}{z^{y+1}}\left(1-\frac{1}{z}\right)^{-n} e^{-(1-z)t}\\ &= \frac{1}{2\pi i}\int_{0,1} dz\left(1-\frac{1}{z}\right)^{-(n+1)} e^{-(1-z)t} = \text{LHS}.\end{aligned} \tag{4.17}$$

(iii)

$$\begin{aligned}\text{LHS} &= \frac{1}{2\pi i}\int_{0,1} dz(1-\frac{1}{z})^{-n}(-1+z)e^{-(1-z)t}\\ &= \frac{1}{2\pi i}\int_{0,1} dz\frac{1}{z^x}(1-\frac{1}{z})^{-n}e^{-(1-z)t} - \frac{1}{2\pi i}\int_{0,1} dz\frac{1}{z^{x+1}}(1-\frac{1}{z})^{-n}e^{-(1-z)t} = \text{RHS}.\end{aligned} \tag{4.18}$$

□

Proof of Prop. 4.1 It is enough to check the forward equation and the initial conditions. Here we only consider the $N=2$ case in which (4.11) reads

$$G(x_1,x_2;t)=\begin{vmatrix} F_0(x_1-y_1;t) & F_1(x_2-y_2;t) \\ F_{-1}(x_1-y_2;t) & F_0(x_2-y_2;t) \end{vmatrix}. \tag{4.19}$$

For our special case where $p=1, q=0$, (4.3),(4.7), (4.5) read

$$\frac{d}{dt}G(x_1,x_2;t)=G(x_1-1,x_2;t)+G(x_1,x_2-1;t)-2G(x_1,x_2;t), \tag{4.20}$$

$$G(x_1,x_1,t)=G(x_1,x_1+1;t), \tag{4.21}$$

$$G(x_1,x_2;t|y_1,y_2;0)=\delta_{x_1y_1}\delta_{x_2y_2}. \tag{4.22}$$

We check (4.19) satisfies them. First about (4.20), we see

$$\begin{aligned}\frac{d}{dt}G(x_1,x_2;t) &= \begin{vmatrix} \frac{d}{dt}F_0(x_1-y_1;t) & F_1(x_2-y_1;t) \\ \frac{d}{dt}F_{-1}(x_1-y_2;t) & F_0(x_2-y_2;t) \end{vmatrix} + \begin{vmatrix} F_0(x_1-y_1;t) & \frac{d}{dt}F_1(x_2-y_1;t) \\ F_{-1}(x_1-y_2;t) & \frac{d}{dt}F_0(x_2-y_2;t) \end{vmatrix} \\ &= \begin{vmatrix} F_0(x_1-y_1-1;t)-F_0(x_1-y_1;t) & F_1(x_2-y_1;t) \\ F_{-1}(x_1-y_2;t)-F_{-1}(x_1-y_2-1;t) & F_0(x_2-y_2;t) \end{vmatrix} \\ &+ \begin{vmatrix} F_0(x_1-y_1;t) & F_1(x_2-y_1;t)-F_1(x_2-y_1-1;t) \\ F_{-1}(x_1-y_2;t) & F_0(x_2-y_2;t)-F_0(x_2-y_2-1;t) \end{vmatrix} \\ &= G(x_1-1,x_2;t)+G(x_1,x_2-1;t)-2G(x_1,x_2;t). \end{aligned} \tag{4.23}$$

For (4.21), we see

$$\begin{aligned} G(x_1,x_1;t) &= \begin{vmatrix} F_0(x_1-y_1;t) & F_1(x_1-y_1;t) \\ F_{-1}(x_1-y_2;t) & F_0(x_1-y_2;t) \end{vmatrix} \\ &= \begin{vmatrix} F_0(x_1-y_1;t) & F_1(x_1-y_1;t)-F_0(x_1-y_1;t) \\ F_{-1}(x_1-y_2;t) & F_0(x_1-y_2;t)-F_{-1}(x_1-y_2;t) \end{vmatrix} \\ &= G(x_1,x_1+1;t) \end{aligned} \tag{4.24}$$

where in the second last equality we used (4.13).

About the initial condition, one sees

$$G(x_1,x_2;t=0)=\begin{vmatrix} F_0(x_1-y_1;0) & F_1(x_2-y_2;0) \\ F_{-1}(x_1-y_2;0) & F_0(x_2-y_2;0) \end{vmatrix} = \begin{vmatrix} \delta_{x_1y_1} & \sum_{z=x_2}^{\infty}\delta_{x_2y_1} \\ \delta_{x_1y_2}-\delta_{x_1,y_2+1} & \delta_{x_2y_2} \end{vmatrix}. \tag{4.25}$$

Since $x_2 \ge y_2 > y_1$, the second term is zero.
Using these one can show that the determinant in (4.11) satisfies (4.3),(4.7), (4.5), i.e. it gives the transition probability for TASEP.

□

4.3 ASEP case [7]

For ASEP with general p, q values, the transition probability becomes rather complicated. First we introduce a notation.

Definition 4.3. *An inversion in a permutation σ is an ordered pair $\{\sigma(i), \sigma(j)\}$ in which $i < j$ and $\sigma(i) < \sigma(j)$.*

$$S_{i,j} = -\frac{p + q\xi_i\xi_j - \xi_i}{p + q\xi_i\xi_j - \xi_j} \tag{4.26}$$

$$A_\sigma = \prod\{S_{i,j} : i, j \text{ is an inversion in } \sigma\} \tag{4.27}$$

A_σ can also be written as

$$A_\sigma = \mathrm{sgn}\sigma \frac{\prod_{i<j}(p + q\xi_{\sigma(i)}\xi_{\sigma(j)} - (p+q)\xi_{\sigma(i)})}{\prod_{i<j}(p + q\xi_i\xi_j - (p+q)\xi_i)}. \tag{4.28}$$

For ASEP, the transition probability is given by [7]

Proposition 4.4.

$$G(x_1, \ldots, x_N; t|y_1, \ldots, y_N; 0) = \sum_{\sigma\in S_N} \int_{C_r} \cdots \int_{C_r} d\xi_1 \ldots d\xi_N A_\sigma \prod_{i=1}^N \xi_{\sigma(i)}^{x_i - y_{\sigma(i)} - 1} e^{\sum_{i=1}^N \epsilon(\xi_i)t} \tag{4.29}$$

where C_r is a contour enclosing the origin anticlockwise with a radius small enough that all the poles in A_σ are not included in C_r. $\epsilon(\xi)$ is defined in (2.19).

The proof is given in [7]. Here to illustrate how it works, we consider the $N = 2$ case. For $N = 2$, the LHS of (4.29) reads

$$G(x_1, x_2; t|y_1, y_2; 0) = \int_{C_r}\int_{C_r} d\xi_1 d\xi_2 \xi_1^{-y_1-1}\xi_2^{-y_2-1}(\xi_1^{x_1}\xi_2^{x_2} + A_{21}\xi_2^{x_1}\xi_1^{x_2})e^{(\epsilon(\xi_1)+\epsilon(\xi_2))t} \tag{4.30}$$

with

$$A_{21} = -\frac{p + q\xi_1\xi_2 - (p+q)\xi_2}{p + q\xi_1\xi_2 - (p+q)\xi_1}. \tag{4.31}$$

We check this satisfies the forward equation (4.3), the boundary condition (4.7) and the initial condition (4.5). The forward equation is easily checked using the expression of $\epsilon(\xi)$. Next we check (4.7). We see

$$\begin{aligned} &pG(x_1, x_1; t) + qG(x_1 + 1, x_1 + 1; t) - (p+q)G(x_1, x_1 + 1; t) \\ &= \int_{C_r}\int_{C_r} d\xi_1 d\xi_2 \xi_1^{x_1-y_1-1}\xi_2^{x_2-y_2-1} e^{(\epsilon(\xi_1)+\epsilon(\xi_2))t} \\ &\quad\times \{p(1 + A_{21}) + q\xi_1\xi_2(1 + A_{21}) - (p+q)(\xi_2 + A_{21}\xi_1)\}. \end{aligned} \tag{4.32}$$

Here the factor in the parenthesis is seen to be zero due to (4.31). Finally we check the initial condition (4.5). Setting $t = 0$ in (4.30), we get

$$G(x_1, x_2, 0) = \int_{C_r}\int_{C_r} d\xi_1 d\xi_2 \xi_1^{-y_1-1}\xi_2^{-y_2-1}\left(\xi_1^{x_1}\xi_2^{x_2} + A_{21}\xi_2^{x_1}\xi_1^{x_2}\right). \tag{4.33}$$

One can easily see that the first term gives $\delta_{x_1y_1}\delta_{x_2y_2}$. So we want to show that the second term is zero, i.e.,

$$\int_{C_r}\int_{C_r} d\xi_1 d\xi_2 \frac{p + q\xi_1\xi_2 - (p+q)\xi_2}{p + q\xi_1\xi_2 - (p+q)\xi_1}\xi_2^{x_1-y_2-1}\xi_1^{x_2-y_1-1} = 0. \tag{4.34}$$

We change the integration variable from ξ_2 to η by

$$\eta = \xi_1\xi_2. \tag{4.35}$$

The integration is over a circle of radius r^2. Then the LHS of (4.34) is

$$\begin{aligned}
&\int_{C_r} d\xi_1 \int_{C_{r^2}} \xi_1 d\eta \frac{p + q\eta - (p+q)\eta/\xi_1}{p + q\eta - (p+q)\xi_1}\left(\frac{\eta}{\xi_1}\right)^{x_1-y_2-1}\xi_1^{x_2-y_1-1} \\
&= \int_{C_r} d\xi_1 \int_{C_{r^2}} d\eta \frac{p + q\eta - (p+q)\eta/\xi_1}{p + q\eta - (p+q)\xi_1}\left(\frac{\eta}{\xi_1}\right)^{x_1-y_2-1}\eta^{x_1-y_2-1}\xi_1^{x_2-x_1+y_2-y_1-1}
\end{aligned} \tag{4.36}$$

Let us consider the ξ_1 integration first. Since $x_2 - x_1 + y_2 - y_1 - 1 \geq 1$, there is no pole at $\xi_1 = 0$. In addition, for small enough r, the denominator is bounded away from zero since $|\xi_1| \leq r, |\eta| = r^2$.

It is interesting that the full transition probability (without any approximation) is obtained by choosing the contours appropriately. More surprisingly, though the formula (4.29) looks rather involved, it can be used to study the current fluctuations of the ASEP [7,20].

5 Random matrix theory [8, 9, 11, 12]

It turns out that the distribution of the current for the ASEP is very much related to the theory of random matrices. In fact for the TASEP with step initial condition, the distribution is equivalent to the Laguerre unitary ensemble.

5.1 One matrix case

In random matrix theory, Gaussian ensembles play a prominent role.

Definition 5.1. *(Gaussian ensembles)*

In Gaussian ensembles, the measure for $N \times N$ matrix H is given in the form,

$$P(H)dH = \frac{1}{Z}e^{-\frac{\beta}{2}\mathrm{Tr}H^2}dH, \quad \beta = 1, 2, 4. \tag{5.1}$$

For GOE($\beta = 1$, Gaussian orthogonal ensemble(GOE)*), H is taken to be a real symmetric matrix and the measure dH is $dH = \prod_{j=1}^N dH_{jj} \prod_{j<l} dH_{jl}$. For GUE($\beta = 2$,* Gaussian unitary ensemble(GUE)*), H is taken to be a hermitian matrix and the measure dH is $dH = \prod_{j=1}^N dH_{jj} \prod_{j<l} dH_{jl}^R \prod_{j<l} dH_{jl}^I$ where H_{jl}^R and H_{jl}^I denotes the real and imaginary part of H_{jl} respectively. GSE($\beta = 4$) means the* Gaussian symplectic ensemble(GSE)*). For a precise definition see a reference.*

For Guassian ensembles, the joint eigenvalue density can be written down explicitly.

Proposition 5.2. *The probability density of eigenvalues, $x_i, 1 \le i \le N$, ($x_i \le x_{i+1}, 1 \le i \le N-1$) is*

$$P(x_1, \ldots, x_N) = \frac{1}{Z} \prod_{j<l} |x_l - x_j|^\beta \prod_{j=1}^N e^{-\frac{\beta}{2} x_j^2}, \quad \beta = 1, 2, 4. \tag{5.2}$$

For a proof see for instance [10, 12].

By using this joint distribution one can in principle study all properties of the eigenvalues of Gaussian ensembles.

In our discussions, the probabilistic properties of the largest eigenvalue are important. From the above joint distribution, one finds that the distribution function of the largest eigenvalue of GUE can be written in the following N fold integral.

Corollary 5.3. *The distribution function of the largest eigenvalue $x_{\max}$*

$$\mathbb{P}_{N2}[x_{\max} \le u] = \frac{1}{Z} \int_{(-\infty,u]^N} \prod_{1 \le j < l \le N} |x_l - x_j|^\beta \prod_{j=1}^N e^{-\frac{\beta}{2} x_j^2} dx_1 \cdots dx_N. \tag{5.3}$$

We are interested in the large N asymptotics of this quantity. This N fold integral expression is not very suited for doing this. We rewrite it into the Fredholm determinant. For the moment let us focus on the $\beta = 2$ case.

Let functions $\Phi_j, \Psi_j : \mathbb{R} \to \mathbb{R}, 0 \le j \le N-1$, s.t. $\int |\Phi_i(x)\Psi_j(x)| dx < \infty, 0 \le i, j \le N-1$

Proposition 5.4. [14] $g \in L^\infty(\mathbb{R}), \mathrm{supp}(g) \subset B(\subset \mathbb{R})$

$$\frac{1}{Z} \int dx_1 \cdots dx_N \det\left(\Phi_j(x_{l+1})\right)_{0 \le j,l \le N-1} \det\left(\Psi_j(x_{l+1})\right)_{0 \le j,l \le N-1} \prod_{j=1}^N (1 + g(x_j))$$
$$= \det(1 + \chi_B K g \chi_B)_{L^2(\mathbb{R})} \tag{5.4}$$

Here det *on RHS is the Fredholm determinant is defined by*

$$\det(1 + \chi_B K g \chi_B)_{L^2(\mathbb{R})} = \sum_{n=0}^\infty \frac{1}{n!} \int_{B^n} \prod_{j=1}^n g(x_j) \det(K(x_i, x_j))_{1 \le i,j \le n} dx_1 \cdots dx_n \tag{5.5}$$

and the kernel $K(x_1, x_2)$ is given by

$$K(x_1, x_2) = \sum_{j,l=0}^{N-1} \Psi_j(x_1)(A^{-1})_{jl} \Phi_l(x_2). \tag{5.6}$$

with

$$A = \{A_{jl}\}_{0\le j,l\le N-1}, \quad A_{jl} = \int \Phi_j(x)\Psi_l(x)dx \tag{5.7}$$

(Assume $\det A \neq 0$*).*

For the proof we use

Proposition 5.5. *(Heine identity)*

$$\begin{aligned} &\int \det\left(\Phi_j(x_{l+1})\right)_{0\le j,l\le N-1} \det\left(\Psi_j(x_{l+1})\right)_{0\le j,l\le N-1} dx_1\cdots dx_N \\ &= N!\det\left(\int \Phi_j(x)\Psi_l(x)dx\right)_{0\le j,l\le N-1}. \end{aligned} \tag{5.8}$$

This is seen as follows

$$\begin{aligned} &\det\left(\int \Phi_j(x)\Psi_l(x)dx\right)_{0\le j,l\le N-1} = \int dx_1\cdots dx_N \det\left(\Phi_i(x_j)\Psi_j(x_i)\right) \\ &= \int dx_1\cdots dx_N \prod_i \Phi_i(x_i)\det\left(\Psi_j(x_i)\right) = \int dx_1\cdots dx_N \prod_i \Phi_i(x_{\sigma(i)})\det\left(\Psi_j(x_{\sigma(i)})\right) \\ &= \int dx_1\cdots dx_N \mathrm{sgn}\sigma \prod_{\mathrm{i}} \Phi_{\mathrm{i}}(\mathrm{x}_{\sigma(\mathrm{i})})\det\left(\Psi_{\mathrm{j}}(\mathrm{x}_{\mathrm{i}})\right) \\ &= \frac{1}{N!}\int dx_1\cdots dx_N \sum_\sigma \mathrm{sgn}\sigma \prod_{\mathrm{i}} \Phi_{\mathrm{i}}(\mathrm{x}_{\sigma(\mathrm{i})})\det\left(\Psi_{\mathrm{j}}(\mathrm{x}_{\mathrm{i}})\right) \\ &= \int dx_1\cdots dx_N \det\left(\Phi_i(x_j)\right)\det\left(\Psi_i(x_j)\right). \end{aligned} \tag{5.9}$$

By using the above proposition, one can show

Proposition 5.6.

$$\mathbb{P}_{N2}[x_{\max} \le u] = \det(1 - \chi_u K_{N2}\chi_u) \tag{5.10}$$

where

$$K_{N2}(x_1,x_2) = e^{-\frac{x_1^2+x_2^2}{2}} \sum_{n=0}^{N-1} \frac{H_n(x_1)H_n(x_2)}{\sqrt{\pi}2^n n!}. \tag{5.11}$$

Here $H_n(x)$ *is the* n *th* Hermite *polynomial*

$$H_n(x) = (-1)^n e^{x^2}\frac{d^n}{dx^n}e^{-x^2}. \tag{5.12}$$

Now it has become easier to consider the large N asymptotics. The basic asymptotics we need is that of the Hermite polynomials [13]. The Airy function $\mathrm{Ai}(x)$ is defined by

$$\mathrm{Ai}(x) = \frac{1}{2\pi}\int_{-\infty+i\epsilon}^{+\infty+i\epsilon} e^{ixz+\frac{iz^3}{3}}dz, \ \ \epsilon > 0. \tag{5.13}$$

Let us write

$$\psi_n(x) = \frac{H_n(x)e^{-x^2/2}}{\pi^{1/4}\sqrt{2^n n!}}. \tag{5.14}$$

Then for the scaling $n = N - N^{1/3}\lambda, x = \sqrt{2N} + \frac{\xi}{\sqrt{2}N^{1/6}}$, we have

$$\psi_N(x) \sim 2^{1/4} N^{-1/12} \mathrm{Ai}(\xi + \lambda). \tag{5.15}$$

To describe the results we introduce a kernel and the corresponding distribution.

Definition 5.7. *The* Airy *kernel* $K_2(x, y)$ *is defined by*

$$K_2(x, y) = \frac{\mathrm{Ai}(x)\mathrm{Ai}'(y) - \mathrm{Ai}(y)\mathrm{Ai}'(x)}{x - y}. \tag{5.16}$$

The GUE Tracy-Widom *distribution* $F_2(s)$ *[15] is defined by*

$$F_2(s) = \det(1 - \chi_s K_2 \chi_s)_{L^2(\mathbb{R})}. \tag{5.17}$$

This is the limiting distribution for the appropriately scaled largest eigenvalue in GUE.

Theorem 5.8.

$$\lim_{N\to\infty} \mathbb{P}_{N2}\left[(x_{\max} - \sqrt{2N})\sqrt{2}N^{1/6} \le s\right] = F_2(s). \tag{5.18}$$

This follows from

Proposition 5.9.

$$\lim_{N\to\infty} \frac{1}{\sqrt{2}N^{1/6}} K_{N2}\left(\sqrt{2N} + \frac{\xi_1}{\sqrt{2}N^{1/6}}, \sqrt{2N} + \frac{\xi_2}{\sqrt{2}N^{1/6}}\right) = K_2(\xi_1, \xi_2), \tag{5.19}$$

which in turn is a consequence of (5.15).

So far our discussions are only for the GUE. One can generalize the discussions to other two cases, GOE,GSE. The limiting distribution is denoted by F_1, F_4

Theorem 5.10.

$$\lim_{N\to\infty} \mathbb{P}_{N\beta}\left[(x_{\max} - \sqrt{2N})\sqrt{2}N^{1/6} \le s\right] = F_\beta(s), \quad \beta = 1, 4. \tag{5.20}$$

5.2 Dyson's Brownian motion

Let us consider a time dependent random matrix $H = H(t)$ of size n of the form,

$$H(t) = \begin{bmatrix} B_{11}(t) & \frac{1}{\sqrt{2}}(B_{12}^{(R)}(t) + iB_{12}^{(I)}(t)) & \cdots & \frac{1}{\sqrt{2}}(B_{1n}^{(R)}(t) + iB_{1n}^{(I)}(t)) \\ \frac{1}{\sqrt{2}}(B_{21}^{(R)}(t) - iB_{21}^{(I)}(t)) & B_{22}(t) & \cdots & \frac{1}{\sqrt{2}}(B_{2n}^{(R)}(t) + iB_{2n}^{(I)}(t)) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\sqrt{2}}(B_{n1}^{(R)}(t) - iB_{n1}^{(I)}(t)) & \frac{1}{\sqrt{2}}(B_{n2}^{(R)}(t) - iB_{n2}^{(I)}(t)) & \cdots & B_{nn}(t) \end{bmatrix}, \tag{5.21}$$

where $B_{jj}, 1 \le j \le n, B^{(R)}_{jk} = B^{(R)}_{kj}, B^{(I)}_{jk} = B^{(I)}_{kj}, 1 \le j < k \le n$ are independent Ornstein-Uhlenbeck processes. Suppose the initial measure is given by (5.1). The joint probability density of $H_n = H(t_n), 0 \le n \le M$ at $M+1$ times $t_n, 0 \le n \le M$ is

$$P(H_0, \dots, H_M) = \frac{1}{Z}\exp\left[-\frac{\beta}{2}\mathrm{Tr}H_0^2\right]\prod_{n=0}^{M-1}\exp\left[-\frac{\beta}{2}\mathrm{Tr}\frac{\left(H_{n+1} - e^{-(t_{n+1}-t_n)}H_n\right)^2}{1 - e^{-2(t_{n+1}-t_n)}}\right]. \tag{5.22}$$

As in the case of the one-matrix case, we are primarily interested in the dynamics of the eigenvalues.

Set $x^n_i = x_i(t_n), 1 \le i \le n, 0 \le n \le M$. Probability density of $\underline{x} = \{x^n_i, 1 \le i \le n, 0 \le n \le M\}$ is given by

$$P_{N2}(\underline{x}) = \frac{1}{Z}\Delta(x^0)\prod_{n=0}^{M-1}\det(\phi^{(t_n,t_{n+1})}_{N2}(x^n_i, x^{n+1}_j))_{1\le i,j\le N}\Delta(x^M), \tag{5.23}$$

where $\Delta(x^n) = \prod_{1\le j<k\le N}(x^n_k - x^n_j), n = 0, M$,

$$\phi^{(t_1,t_2)}_{N2}(x_1, x_2) = \frac{e^{-\frac{1}{2}(t_2-t_1)}}{\sqrt{\pi(1 - e^{-2(t_2-t_1)})}}\exp\left[-\frac{(x_2 - e^{-(t_2-t_1)}x_1)^2}{1 - e^{-2(t_2-t_1)}}\right]. \tag{5.24}$$

By an extension of the arguments to get (5.6), the joint distribution function of the largest eigenvalue in the GUE Dyson's Brownian motion is given by the Fredholm determinant with the kernel,

$$K_{N2}(t_1, x_1; t_2, x_2) = e^{-x_2^2}\sum_{n=0}^{N-1}\frac{H_n(x_1)H_n(x_2)}{\sqrt{\pi}2^n n!}e^{-(n+\frac{1}{2})(t_2-t_1)} - \phi^{(t_1,t_2)}_{N2}(x_1, x_2)1_{t_1<t_2}. \tag{5.25}$$

Next we consider the asymptotics of the Dyson's BM.

Definition 5.11. [16] *The extended* Airy *kernel* $K_2(\tau_1, \xi_1; \tau_2, \xi_2)$

$$K_2(\tau_1, \xi_1; \tau_2, \xi_2) = \int_0^\infty d\lambda e^{-\lambda(\tau_1-\tau_2)}\mathrm{Ai}(\xi_1 + \lambda)\mathrm{Ai}(\xi_2 + \lambda) - \phi^{(\tau_1,\tau_2)}_2(\xi_1, \xi_2)1_{\tau_1<\tau_2}. \tag{5.26}$$

with

$$\phi^{(\tau_1,\tau_2)}_2(\xi_1, \xi_2) = \frac{1}{\sqrt{4\pi(\tau_2 - \tau_1)}}\exp\left[-\frac{(\xi_2 - \xi_1)^2}{4(\tau_2 - \tau_1)} - \frac{1}{2}(\tau_2 - \tau_1)(\xi_1 + \xi_2) + \frac{1}{12}(\tau_2 - \tau_1)^3\right]. \tag{5.27}$$

The Airy$_2$ *process,* $\mathcal{A}_2$ *is defined to be a process with* m *point distributions at* $\tau_1 < \tau_2 < \ldots < \tau_m$ *is given by the* Fredholm *determinant,*

$$\mathbb{P}\left(\bigcap_{k=1}^m\{\mathcal{A}_2(\tau_k) \le s_k\}\right) = \det(1 - \chi_s K_2 \chi_s)_{L^2(\{\tau_1,\ldots,\tau_m\}\times\mathbb{R})} \tag{5.28}$$

where $\chi_s(\tau_k, x) = 1_{(s_k,\infty)}(x)$.

Define the scaled largest eigenvalue

$$A_{N2}(\tau) = \left(x_{\max}\left(\frac{\tau}{N^{1/3}}\right) - \sqrt{2N}\right)\sqrt{2}N^{1/6} \tag{5.29}$$

Theorem 5.12. *For finite dimensional distributions*

$$\lim_{N\to\infty} A_{N2} = \mathcal{A}_2. \tag{5.30}$$

6 Current fluctuations of ASEP [17]

In this section, we study the distribution of the integrated current in ASEP and see that the random matrix theory in the last section plays a crucial role. In particular for TASEP with step initial condition, the connection is very direct.

6.1 LUE formula for TASEP with step initial condition

Let us consider TASEP ($p = 1, q = 0$) with the step initial condition in which initially at time $t = 0$ all sites $x \leq 0$ are occupied and all other sites $x \geq 1$ are empty. The quantity we are interested in is the number of particles which crossed the edge (or bond) between sites 0 and 1 up to time t, which we denote by $N(t)$. Then the distribution function of this can be written in an N fold integral.

Proposition 6.1.

$$\mathbb{P}[N(t) \geq N] = \frac{1}{Z'_{N2}} \int_{[0,t]^N} \prod_{1\leq j<k\leq N} (x_j - x_k)^2 \prod_{j=1}^{N} e^{-x_j} dx_1 \dots dx_N. \tag{6.1}$$

Z'_{N2} *is a normalization.*

The similarity of this expression to (5.3) is obvious. In fact the same asymptotic analysis can be applied to (6.1) and as a result we have

Theorem 6.2.

$$\lim_{t\to\infty} \mathbb{P}\left[\frac{\frac{t}{4} - N(t)}{2^{-4/3}t^{1/3}} \leq s\right] = F_2(s) \tag{6.2}$$

This says that the fluctuation of the integrated current of the TASEP with step initial condition is in the appropriate scaling limit the same as that of the largest eigenvalue distribution of GUE. Remembering the simple definition of ASEP (in particular TASEP), this connection is totally unexpected. Below we explain how we can get the formula (6.1).

6.2 Directed polymer [17, 18]

First we follow the original argument of Johansson. In this subsection we consider the discrete time TASEP with parallel update. Each particle tries to do the random walk with hopping probability to the right $1-q$ and remains at the same site with probability q. Here we are mainly interested in the step initial condition and the number of particles which jumped from 0 to 1 up to time n is denoted by $N(n)$.

Let us consider a matrix $W = \{W_{i,j}\}$, in which the (i,j) element $W_{i,j}$ represents the waiting time of the jth particle before making the ith hop after the target site becomes empty. We call this the "waiting time" table in which the whole time evolution of the particles is encoded. For example the matrix looks like

$$w = \begin{bmatrix} 1 & 1 & 1 & 3 & \cdots \\ 2 & 2 & 1 & 0 & \\ 1 & 0 & 0 & 1 & \\ 1 & 2 & 1 & 0 & \\ \vdots & & & & \ddots \end{bmatrix}. \tag{6.3}$$

Up to now our explanation is for a sample. If one remembers the TASEP dynamics, $\{W_{i,j}\}_{i,j\geq 1}$ is i.i.d. random variables with each of them being geometrically distributed with parameter q: $W_{i,j} \sim \mathrm{Ge}(q)$. We define

$$G(M,N) = \max_{\pi\in\Pi_{M,N}} \{ \sum_{(i,j)\in\pi} W_{i,j} \}. \tag{6.4}$$

In the matrix representation as in (6.3), π is a down/right path, i.e., it starts from $(1,1)$ and goes either down or right until it reaches (M,N). $\Pi_{M,N}$ is the collection of all such paths. One can regard $W_{i,j}$ as representing a value of a potential energy at a position (i,j). Then the quantity (6.4) can be considered as a problem of the zero temperature directed polymer in random medium. Then one can confirm that $G(M,N)+M+N-1$ represents the arrival time of N th particle at site $M-N+1$ and hence it holds

Proposition 6.3. $\mathbb{P}[N(n)\geq N] = \mathbb{P}[G(N,N)+2N-1\leq n]$.

Therefore our problem of the integrated current $N(n)$ for TASEP is now mapped to that of $G(N,N)$ with iid W_{ij}'s. For a given N we can restrict our attention to the $N\times N$ submatrix $\{W_{i,j}\}_{1\leq i,j\leq N}$ of W. For instance, from the 3×3 submatrix of w,

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 2 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \tag{6.5}$$

one sees that $G(3,3)=6$ with the maximizing path, $(1,1)\to(2,1)\to(2,2)\to(2,3)\to(3,3)$.

One can further map the problem to another. We use a bijection known as the Robinson-Schensted-Knuth(RSK) algorithm between an $\mathbb{N}$ matrix and a pair (P,Q) of semistandard Young tableaux(SSYT). For SSYT and the RSK algorithm, see for instance [42, 43]. The pair of SSYTs corresponding to (6.5) is

$$P = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 2 & 3 \\ \hline 2 & 2 \\ \cline{1-2} 3 \\ \cline{1-1} \end{array}, \quad Q = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 2 & 2 & 2 \\ \hline 2 & 2 \\ \cline{1-2} 3 \\ \cline{1-1} \end{array}.$$

In the language of SSYT, the $G(N,N)$ is given by

$$G(N,N) = \lambda_1 \tag{6.6}$$

where λ_1 is the length of the first row of P. This is easily seen as a property of the RSK algorithm. For our example, $\lambda_1 = 6$ is the same as $G(3,3)$. Therefore the problem of $N(t)$ is reduced to a combinatorial problem of SSYTs. Using this connection we have

$$\mathbb{P}[G(N,N) + 2N - 1 \leq t] = (1-q)^{N^2} \sum_{\lambda:\lambda_1 \leq t-2N+1} q^{|\lambda|} L(\lambda, N)^2 \tag{6.7}$$

Here $L(\lambda, N)$ is the number of SSYT of shape λ with entries from $\{1, 2, \ldots, N\}$, for which there is a formula,

$$L(\lambda, N) = \prod_{1 \leq i < j \leq N} \frac{\lambda_i - \lambda_j + j - i}{j - i}. \tag{6.8}$$

Rewriting (6.7) using the formula (6.8) and $h_j = \lambda_1 - j + 1$, one obtains

$$\mathbb{P}[G(N,N) + 2N - 1 \leq t] = \frac{1}{Z} \sum_{h_j=-N+1}^{t-2N+1} \prod_{1 \leq j < l \leq N} (h_j - h_l)^2 \prod_{j=1}^{N} q^{h_j}. \tag{6.9}$$

Here and in the following the symbol Z is used to represent a normalization constant. For the present case it is

$$Z = \frac{q^{\frac{1}{2}N(N-1)}}{(1-q)^{N^2}} \prod_{j=1}^{N} j!(j-1)!. \tag{6.10}$$

The expression of the RHS of (6.9) has a strong similarity with (6.1). In fact in the continuous time limit of TASEP, $W_{i,j}$ becomes an exponential random variable with parameter one and (6.9) reduces to (6.1).

6.3 Using transition probability

The original combinatorial argument by Johansson is ingenious and gives a new perspective to the problem but requires a knowledge of different field and in a sense indirect. There is a more direct approach to the same formula using the transition probability we studied in section 4.

Let us consider the TASEP in which the initial positions of the N rightmost particles are $y_j, 1 \leq j \leq N$. Then one can show

Proposition 6.4. *Under the condition $M > y_N - y_1$,*

$$\begin{aligned}
&\mathbb{P}[X_1(t) \geq y_1 + M] \\
&= \sum_{y_1+M \leq x_1 < x_2 < \cdots < x_N} G(x_1, x_2, \cdots, x_N; t | y_1, y_2, \cdots, y_N; 0) \\
&= \frac{1}{\prod_{j=1}^{N} j!} \int_{[0,t]^N} dt_1 \cdots dt_N \prod_{1 \leq j < k \leq N} (t_k - t_j) \det\left(F_{-j+1}(y_1 - y_j + M - 1; t_{N-k+1})\right).
\end{aligned} \tag{6.11}$$

The step initial condition corresponds to setting $y_j = -j+1, j = 1, \cdots, N$. In this case (6.11) reduces to (6.1). Here we only see how the computation proceeds for $N = 2$;

$$
\begin{aligned}
\mathbb{P}[N(t) \geq 2] &= \sum_{1 \leq x_1 < x_2} G(x_1, x_2; t | y_1 = -1, y_2 = 0; 0) \\
&= \begin{vmatrix} F_1(2;t) & F_2(3;t) \\ F_0(1;t) & F_1(2;t) \end{vmatrix} = \int_0^t dt_2 \int_0^t ds \begin{vmatrix} F_0(1;t_2) & F_1(2;s) \\ F_{-1}(0;t_2) & F_0(1;s) \end{vmatrix} \\
&= \int_0^t dt_2 \int_0^t ds \int_0^s dt_1 \begin{vmatrix} F_0(1;t_2) & F_0(1;t_1) \\ F_{-1}(0;t_2) & F_{-1}(0;t_1) \end{vmatrix} \\
&= \int_0^t dt_2 \int_0^t ds (t - t_1) \begin{vmatrix} F_0(1;t_2) & F_0(1;t_1) \\ F_{-1}(0;t_2) & F_{-1}(0;t_1) \end{vmatrix} \\
&= \frac{1}{2} \int_0^t dt_2 \int_0^t ds (t_2 - t_1) \begin{vmatrix} F_0(1;t_2) & F_0(1;t_1) \\ F_{-1}(0;t_2) & F_{-1}(0;t_1) \end{vmatrix} \\
&= \frac{1}{2} \int_0^t dt_2 \int_0^t ds (t_2 - t_1) \begin{vmatrix} F_0(1;t_2) & F_0(1;t_1) \\ F_0(0;t_2) & F_0(0;t_1) \end{vmatrix} \\
&= \frac{1}{2} \int_0^t dt_2 \int_0^t ds (t_2 - t_1) \begin{vmatrix} t_2 e^{-t_2} & t_1 e^{-t_2} \\ e^{-t_2} & e^{-t_1} \end{vmatrix} \\
&= \frac{1}{2} \int_0^t dt_2 \int_0^t ds (t_2 - t_1)^2 e^{-t_1 - t_2}.
\end{aligned}
\tag{6.12}
$$

The computation can be generalized to arbitrary N and in this way one can arrive at (6.1).

6.4 ASEP case [7, 19, 20]

After the work of Johansson a natural question was if one can study the general ASEP in which particles hop in both directions. This had remained unsolved for several years until when Tracy and Widom succeeded in manipulating the transition probability to study the distribution of a particle position. We do not intend to explain the details but show a few key formulas.

Here we change the notation s.t. the average current is to the left ($q > p$) with $\tau = p/q$ and the step initial condition is s.t. the all sites on $x \geq 1$ are occupied by particles. The starting point is the transition probability in (4.29). By summing them up appropriately, one can find

Proposition 6.5.

$$
\begin{aligned}
\mathbb{P}[x_m(t) \leq x] = (-1)^m \sum_{k \geq m} \frac{1}{k!} \binom{k-1}{k-m}_{\tau} p^{(k-m)(k-m+1)/2} q^{km+(k-m-1)/2} \\
\times \int_{C_R} \cdots \int_{C_R} \prod_{i \neq j} \frac{\xi_j - \xi_i}{p + q\xi_i\xi_j - \xi_i} \prod_i \frac{1}{(1-\xi_i)(q\xi_i - p)} \prod_i \left(\xi_i^x e^{\epsilon(\xi_i)t} \right) d\xi_1 \ldots d\xi_N
\end{aligned}
\tag{6.13}
$$

In the proof one needs a few identities such as

Lemma 6.6.

$$\sum_{\sigma\in S_N}\operatorname{sgn}(\sigma)\left(\prod_{i<j}(p+\xi_{\sigma(i)}\xi_{\sigma(j)}-\xi_{\sigma(i)})\right)\frac{\xi_{\sigma(2)}\xi_{\sigma(3)}^2\cdots\xi_{\sigma(N)}^{N-1}}{(1-\xi_{\sigma(1)}\cdots\xi_{\sigma(N)})\cdots(1-\xi_{\sigma(N-1)}\xi_{\sigma(N)})(1-\xi_{\sigma(N)})}$$
$$=p^{N(N-1)/2}\frac{\prod_{i<j}(\xi_j-\xi_i)}{\prod_j(1-\xi_j)}. \tag{6.14}$$

One can rewrite the quantity (6.13) as a contour integral of a Fredholm determinant. Let us set

$$(\lambda;\tau)_m=(1-\lambda)(1-\lambda\tau)\cdots(1-\lambda\tau^{m-1}), \tag{6.15}$$
$$K(\xi,\xi')=\frac{\xi^x e^{\epsilon(\xi)t}}{p+q\xi\xi'-\xi}. \tag{6.16}$$

Then it is fairy easy to see

Proposition 6.7.
$$\mathbb{P}[x_m(t)\le x]=\int\frac{\det(1-\lambda qK)}{(\lambda;\tau)_m}\frac{d\lambda}{\lambda} \tag{6.17}$$

where the integral is taken over a large circle.

This Fredholm determinant expression is not very suitable for asymptotic analysis. Let us define

$$\varphi_\infty(\eta)=\frac{e^{t\eta/(1-\eta)}}{(1-\eta)^x}, \tag{6.18}$$
$$f(\mu,z)=\sum_{k\in\mathbb{Z}}\frac{\tau^k}{1-\tau^k\mu}z^k, \tag{6.19}$$
$$J(\mu;\eta,\eta')=\int_{C_1}\frac{\varphi_\infty(\zeta)}{\varphi_\infty(\eta')}\frac{\zeta^m}{(\eta')^{m+1}}\frac{d(\mu,\zeta/\eta')}{\zeta-\eta}d\zeta. \tag{6.20}$$

Here η,η' are on a circle with center zero and radius $r\in(\tau,1)$. C_1 is a circle with center zero and radius $r\in(1,r/\tau)$. Then one can show

Proposition 6.8.

$$\mathbb{P}\left[x_m\left(\frac{t}{q-p}\right)\le x\right]=\int_{C_0}\prod_{k=0}^{\infty}(1-\mu\tau^k)\det(1+\mu J(\mu))\frac{d\mu}{\mu} \tag{6.21}$$

C_1 is a circle with center zero and radius $r\in(\tau,1)$.

Using this expression, one can apply the steepest descent method. Define

$$\sigma=m/t,\ c_1=-1+2\sqrt{\sigma},\quad c_2=\sigma^{-1/6}(1-\sqrt{\sigma})^{2/3} \tag{6.22}$$

The result is

Theorem 6.9.
$$\lim_{t\to\infty} \mathbb{P}\left[\frac{x_m(\frac{t}{q-p}) - c_1 t}{c_2 t^{1/3}} \leq x\right] = F_2(s). \tag{6.23}$$

This is of course an expected one from the universality but the fact that it was proved was a big progress.

6.5 Generalizations

6.5.1 Multipoint fluctuations [16, 21, 22]

So far the discussions are only for the one point distributions. By using a multilayer growth model, which can be considered as a generalization of the arguments in section 6.2, one can study the multipoint joint distributions. The measure is written as a product of determinants, which look similar to (5.23).

As a result of this, for the step initial condition, the multipoint fluctuations are described by the Airy_2 process.

6.5.2 Flat Case [23–26]

The one point fluctuation of the alternating initial condition (flat initial condition for surface growth) is described by the GOE TW distribution. It is difficult to study the mutli-point distribution by the same argument as for the step. It turned out that a generalization of the analysis using the transition probability can be applied.

Definition 6.10. *The* Airy_1 *kernel* $K_1(\tau_1, \xi_1; \tau_2, \xi_2)$ *is defined by*

$$\begin{aligned} K_1(\tau_1, \xi_1; \tau_2, \xi_2) &= \mathrm{Ai}(\xi_1 + \xi_2 + (\tau_2 - \tau_1)^2) \exp\left((\tau_2 - \tau_1)(\xi_1 + \xi_2) + \frac{2}{3}(\tau_2 - \tau_1)^3\right) \\ &\quad - \frac{1}{\sqrt{4\pi(\tau_2 - \tau_1)}} \exp\left(-\frac{(\xi_2 - \xi_1)^2}{4(\tau_2 - \tau_1)}\right) 1_{\tau_1 < \tau_2}. \end{aligned} \tag{6.24}$$

The Airy_1 *process* $\mathcal{A}_1$ *is defined by its* m *point joint distributions at* $\tau_1 < \tau_2 < \ldots < \tau_m$ *by the* Fredholm *determinant*

$$\mathbb{P}\left(\bigcap_{k=1}^{m} \{\mathcal{A}_1(\tau_k) \leq s_k\}\right) = \det(1 - \chi_s K_1 \chi_s)_{L^2(\{\tau_1, \ldots, \tau_m\} \times \mathbb{R})} \tag{6.25}$$

This process describes the multi-point fluctuations of the flat surface

7 Gelfand-Tsetlin dynamics [27, 28]

As we have seen, current fluctuations of ASEP is related to a quantity from random matrix theory. For TASEP we have explained two arguments to arrive at the LUE formula. The one is a combinatorial argument and the other is to use the transition probability. Both of them include rather involved mapping of the measure or the multiple integral. Here we show another approach to the LUE formula, in which we first enlarge the system we look at and then realize the TASEP and Dyson's Brownian motion as two different slices of it. In this section we explain this using the diffusion limit of TASEP.

7.1 Diffusion limit of TASEP

We consider a process in which we replace each particle in TASEP by a BM. What we find is the Z process $Z_1 \le Z_2 \le \ldots \le Z_n$ in which Z_1 is a Brownian motion and Z_{j+1} is reflected by Z_j, $1 \le j \le n-1$. More precisely, one has an explicit expression,

$$\begin{aligned} Z_1(t) &= B_1(t), \\ Z_j(t) &= \sup_{0\le s\le t} (Z_{j-1}(s) + B_j(t) - B_j(s)), \quad 2 \le j \le n. \end{aligned} \tag{7.1}$$

The Schütz type formula for the transition probability of this Z process is given by

Proposition 7.1. *[27, 29] Transition density of Z process is*

$$G(x_1, \ldots, x_N; t|y_1, \ldots, y_N; 0) = \det(F_{l-j}(x_l - y_j; t))_{1\le j,l\le N} \tag{7.2}$$

where

$$F_n(x,t) = \frac{i^n}{2\pi} \int_{-\infty}^{\infty} dk \frac{e^{-k^2 t + ikx}}{(k+i\epsilon)^n}. \tag{7.3}$$

Using this one can show an analog of the LUE formula for TASEP.

Proposition 7.2.

$$\Pr[Z_N(t) \le x_0] = \frac{1}{Z_N(t)} \int_{-\infty}^{x_0} dx_1 \cdots \int_{-\infty}^{x_0} dx_N \prod_{1\le i<j\le N} (x_i - x_j)^2 \prod_{i=1}^{N} e^{-\frac{x_i^2}{2t}} \tag{7.4}$$

7.2 Non-colliding BM and Dyson's BM

Let us consider a time dependent random matrix of the form (5.21) where $B_{jj}, 1 \le j \le n, B_{jk}^{(R)} = B_{kj}^{(R)}, B_{jk}^{(I)} = B_{kj}^{(I)}, 1 \le j < k \le n$ are independent Brownian motions.

The stochastic dynamics of the n eigenvalues of H denoted by $X_1 \le X_2 \le \ldots \le X_n$ is described by the stochastic differential equation(SDE),

$$dX_i = dB_i + \sum_{\substack{1\le j\le m \\ j\ne i}} \frac{dt}{X_i - X_j}, \quad 1 \le i \le n, \tag{7.5}$$

where $B_i, 1 \le i \le n$ are independent one dimensional Brownian motions. This is known as Dyson's Brownian motion (DBM). The process satisfies $X_1(t) < X_2(t) < \cdots < X_n(t)$ for all $t > 0$. The process X can be started from the origin, i.e., one can take $X_i(0) = 0, 1 \le i \le n$.

The corresponding Kolmogorov backward equation for Dyson's Brownian motion reads

$$\frac{\partial}{\partial t}p_t^+ = \frac{1}{2}\sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2}p_t^+ + \sum_{i=1}^{n}\sum_{j\neq i}\frac{1}{x_i - x_j}\cdot\frac{\partial p_t^+}{\partial x_i}. \tag{7.6}$$

Set

$$h_n(x) = \prod_{1\le i<j\le n}(x_j - x_i) \tag{7.7}$$

Then (7.6) can be written as

$$\frac{\partial}{\partial t}p_t^+ = \frac{1}{2}\sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2}p_t^+ + \sum_{i=1}^{n}\frac{\partial}{\partial x_i}\log h_n(x)\cdot\frac{\partial p_t^+}{\partial x_i}. \tag{7.8}$$

Dyson's BM can be constructed from noncolliding Brownian motion through Doob's h-transformation using the function (7.7) [31]. By Karlin-McGregor formula, the transition density of non-colliding BM with n particles is

$$p_t(x,x') = \det(\phi_t(x_i, x_j'))_{1\le i,j\le n} \tag{7.9}$$

with

$$\phi_t(x,x') = \frac{1}{\sqrt{2\pi t}}e^{-(x-x')^2/(2t)}. \tag{7.10}$$

This satisfies

$$\frac{\partial}{\partial t}p_t = \frac{1}{2}\sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2}p_t \tag{7.11}$$

and

$$p_t|_{x_i=x_{i+1}} = 0. \tag{7.12}$$

The above h_n in (7.7) is harmonic for this,

$$\int_{-\infty}^{\infty}dx' p_t(x,x')h_n(x') = h_n(x). \tag{7.13}$$

We define the h-transform of $p_t(x,x')$ by

$$p_t^+(x,x') := \frac{h_n(x')}{h_n(x)}p_t(x,x'). \tag{7.14}$$

Then one has

Proposition 7.3. *RHS of (7.14) satisfies (7.6).*

Proof. From the definition (7.14) and (7.11) we have

$$\frac{\partial}{\partial t}p_t^+(x,x') = \frac{1}{2}\frac{h_n(x')}{h_n(x)}\sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2}p_t. \tag{7.15}$$

By a straightforward computation one sees

$$\frac{\partial}{\partial x_i}p_t^+ = -\frac{\partial}{\partial x_i}\log h_n(x)\cdot p_t^+ + \frac{h_n(x')}{h_n(x)}\frac{\partial}{\partial x_i}p_t, \tag{7.16}$$

$$\frac{\partial^2}{\partial x_i^2}p_t^+ = \frac{h_n(x')}{h_n(x)}\frac{\partial^2}{\partial x_i^2}p_t d - 2\frac{\partial}{\partial x_i}\log h_n(x)\frac{\partial}{\partial x_i}p_t^+ + \left\{-\frac{\partial^2}{\partial x_i^2}\log h_n(x) - \left(\frac{\partial}{\partial x_i}\log h_n(x)\right)^2\right\}p_t^+. \tag{7.17}$$

Here one computes

$$-\frac{\partial^2}{\partial x_i^2}\log h_n(x) - \left(\frac{\partial}{\partial x_i}\log h_n(x)\right)^2 = -2\sum_{j,k(\neq i)}\frac{1}{(x_i-x_j)(x_i-x_k)}. \tag{7.18}$$

Noticing

$$\sum_i \sum_{j,k(\neq i)} \frac{1}{(x_i-x_j)(x_i-x_k)} = 0, \tag{7.19}$$

we see that the RHS of (7.14) satisfies (7.8). □

7.3 Interlacing of Dyson's Brownian motion

Let $W_j, 1 \le j \le n$ be the Dyson's BM with n particles starting from the origin. We construct a new process $X_j, 1 \le j \le n+1$ of $n+1$ particles, in which each X_j performs a BM with the conditions that they interlace W_j, i.e. $X_1 \le W_1 \le X_2 \le \ldots \le W_n \le X_{n+1}$. The interlacing is maintained by prescribing that X_j is reflected from W_{j-1} and W_j. Then it holds

Theorem 7.4. *The $X_j, 1 \le j \le n+1$ is Dyson's BM with $n+1$ particles.*

By repeating this interlacing procedure from $n = 1$ to N, one can construct a process of $\frac{N}{2}(N+1)$ particles. The position of particles, $x_i^k \in \mathbb{R}, 1 \le i \le k \le N$ satisfy the constraint $x_i^{k+1} \le x_i^k \le x_{i+1}^k$, which is known as the Gelfand-Tsetlin cone.

Each row is Dyson's BM. On the other hand $x_i^i, 1 \le i \le N$ is the Z process. Using this picture it is clear that the distribution of the nth particle in the Z process is the same as that of the top particle in the n particle Dyson's BM. In other words, we now have a more conceptual understanding of (7.4). The same construction works also for TASEP.

8 KPZ equation [36–40]

8.1 Introduction

In the studies of surface growth phenomena, a prototypical model equation is the Kardar-Parisi-Zhang (KPZ) equation [35]. For $x \in \mathbb{R}, t \ge 0, h(x,t) \in \mathbb{R}$, it reads

$$\partial_t h(x,t) = \tfrac{1}{2}\lambda(\partial_x h(x,t))^2 + \nu\partial_x^2 h(x,t) + \sqrt{D}\eta(x,t)$$

$$
\begin{array}{ccccccccc}
 & & & & x_1^1 & & & & \\
 & & & x_1^2 & & x_2^2 & & & \\
 & & x_1^3 & & x_2^3 & & x_3^3 & & \\
 & \iddots & & & \vdots & & & \ddots & \\
x_1^n & x_2^n & x_3^n & & \cdots & & & x_{n-1}^n & x_n^n
\end{array}
$$

図 1: Gelfand-Tsetlin cone

where $\lambda, \nu, D > 0$ and $\eta(x,t)$ is a Gaussian white noise with $\langle \eta(x,t)\eta(x',t')\rangle = \delta(x-x')\delta(t-t')$. Here $\langle\cdot\rangle$ means the average wrt η. Kardar, Parisi and Zhang applied a dynamical version of the renormalization group analysis to show that the height fluctuations scale like $O(t^{1/3})$ as $t \to \infty$. There have been a lot of (physical) analysis of this equation, but there had not been much information available for the height distribution until recently. In 2010, the first explicit distribution has been identified for a special initial condition called the narrow wedge initial condition. In this section we explain this new development for the height distribution of the KPZ equation.

Before going into the main discussions, we remark that the KPZ equation has a scaling property. If one sets

$$X = \alpha^2 x, \quad T = 2\nu\alpha^4 t, \quad H = \frac{\lambda}{2\nu} h, \tag{8.1}$$

with $\alpha = (2\nu)^{-3/2}\lambda D^{1/2}$, $H(X,T)$ satisfies

$$\frac{\partial}{\partial T} H = \frac{1}{2}\Big(\frac{\partial H}{\partial X}\Big)^2 + \frac{1}{2}\frac{\partial^2}{\partial X^2} H + \eta\,. \tag{8.2}$$

We set $\lambda = 1$, $\nu = \frac{1}{2}$, and $D = 1$ in the following.

8.2 Cole-Hopf solution

The KPZ equation itself is ill-defined as it is due to the irregular behaviors of $h(x)$. A way of defining the equation was proposed by Bertini-Giacomin in [41]. One starts from the stochastic heat equation,

$$\frac{\partial}{\partial t} Z = \frac{1}{2}\frac{\partial^2}{\partial x^2} Z + \eta Z, \tag{8.3}$$

which is a well-defined SDE of Itô-type, and then *defines* the solution to the KPZ equation by

$$h(x,t) = \log Z(x,t). \tag{8.4}$$

8.3 The distribution for narrow wedge

For the stochastic heat equation (8.3), we consider the initial condition,

$$Z(x,0) = \delta(x). \tag{8.5}$$

For the KPZ equation, one might think that this corresponds to the following narrow wedge initial conditions:

$$h(x,0) = -|x|/\delta\,, \quad \delta \ll 1. \tag{8.6}$$

For finite $t > 0$, the macroscopic shape of the surface is given by

$$h(x,t) = \begin{cases} -x^2/2t, & \text{for } |x| \le t/\delta\,, \\ (1/2\delta^2)t - |x|/\delta, & \text{for } |x| > t/\delta. \end{cases} \tag{8.7}$$

Let ξ_t have the probability density,

$$\begin{aligned} \rho_t(s) = & \int_{-\infty}^{\infty} \gamma_t \mathrm{e}^{\gamma_t(s-u)} \exp\big[-\mathrm{e}^{\gamma_t(s-u)}\big] \\ & \times \big(\det(1 - P_u(B_t - P_{\mathrm{Ai}})P_u) - \det(1 - P_u B_t P_u)\big)\mathrm{d}u \end{aligned} \tag{8.8}$$

where $P_{\mathrm{Ai}}(x,y) = \mathrm{Ai}(x)\mathrm{Ai}(y)$, P_u is the projection onto $[u,\infty)$ and the kernel B_t is

$$B_t(x,y) = K_{\mathrm{Ai}}(x,y) + \int_0^\infty \mathrm{d}\lambda (\mathrm{e}^{\gamma_t \lambda} - 1)^{-1} \times \big(\mathrm{Ai}(x+\lambda)\mathrm{Ai}(y+\lambda) - \mathrm{Ai}(x-\lambda)\mathrm{Ai}(y-\lambda)\big)\,. \tag{8.9}$$

Theorem 8.1. *For the KPZ height $h(x,t)$ with the narrow wedege initial condition,*

$$h(x,t) \stackrel{d}{=} -x^2/2t - \tfrac{1}{12}\gamma_t^3 + \gamma_t \xi_t \tag{8.10}$$

Here $\gamma_t = 2^{-1/3}t^{1/3}$.

This is obtained by considering a weakly asymmetric limit of the ASEP. One utilizes the formula (6.21). It requires rather intricate asymptotic analysis.

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