

点過程からの準備

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概要

このノートでは、点過程についていくつかの事実をまとめておく。点過程は、無限次元力学系の「平衡分布」の役割を果たすのもで、有限次元の Birkhoff motion の場合は、Lebesgue 測度に対応する。

0 配置空間・点過程・相関関数

空間 S 内の有限もしくは可算無限個の粒子を表す空間を設定する。 S とは粒子が動き回っている空間で、一般的な枠組みとしては S を Polish 空間とする。Polish 空間とは、完備可分距離空間と位相同型な空間である。「と位相同型な」とは持って回った言い方だが、便利である。例えば、 $(0, 1)$ は通所のユークリッド距離で完備距離空間とはならないが、 \mathbb{R} と位相同型だからポーランド空間となる。確率論は通常ポーランド空間位で設定するのが、一般性と具体性のバランスがうまくとれている。尚、ポーランド空間のボレル可測部分集合は、(部分位相の下で) 再びポーランド空間となる。

ただし、この本では S は \mathbb{R}^d 、 \mathbb{C} もしくは $[0, \infty)$ だけが登場する。後に S の上の粒子の空間である配置空間を考えると、一般に S を局所コンパクトを仮定すると、「大丈夫」だが、Polish 空間にする場合は、少し注意が必要である。Resnick の本では Polish で扱っているが、一般の書籍ではそうでない。

$M(S)$ で S のラドン測度全体を表す。ここでラドン測度とは、可測空間 $(S, \mathcal{B}(S))$ 上の測度でコンパクト集合に対し有限の値を取るものである。ここで $\mathcal{B}(S)$ は S のボレル集合族 (開集合を含む最小の σ 加法族) を表す。 $M(S)$ には、漠位相 (Vague topology) と弱位相 (weak topology) という、次の 2 種類の位相をしばしば考える。

dfn:11
定義 0.1.

$$\lim_{n \rightarrow \infty} \mu_n = \mu \Leftrightarrow \lim_{n \rightarrow \infty} \int f d\mu_n = \int f d\mu \text{ for all } f \in C_b(S) \quad (\text{weak})$$

$$\lim_{n \rightarrow \infty} \mu_n = \mu \Leftrightarrow \lim_{n \rightarrow \infty} \int f d\mu_n = \int f d\mu \text{ for all } f \in C_0(S) \quad (\text{vague})$$

ここで $C_0(S)$ はコンパクト集合の外では 0 となる、連続関数とする。尚、この選択は、 S の位相だけにより、その位相を実現する距離にはよらないので便利である。しかしより詳しい性質を見るとき、距離に依存した細かい概念に変えておいたほうが便利な場合もある。ともかくここでは有限次元の完備距離空間に対してしか使わない。

粒子の空間として、2 種類の空間が自然に考えられる。一つは、すべての粒子を区別する場合で、これは粒子の個数分だけ S の直積をとった空間である。例えば加算無限個の粒子に、順に $1, 2, 3, \dots$ とラベルを付けた場合は、 $S^{\mathbb{N}}$ となる。もう一つは、すべての粒子を区別しない

空間で、配置空間と呼ばれる。粒子を区別した場合ラベル粒子系、また区別しない場合アンラベル粒子系と呼ぶことにする。

より正確に述べる。

空間 S 内の n 個の粒子は S^n の元と扱って差し支えないが、無限粒子系の場合、一般に粒子にラベルを付け S^∞ の元として扱うと平衡状態を表現する確率測度を構成することが難しくなる。そこで個々の粒子を区別しない配置空間を導入する必要がある。

配置空間

^{dfn:21}**定義 0.2.** Polish 空間 S 上の非負整数値ラドン測度 s を配置 (configuration) といい、 S の配置全体 \mathcal{S} に、漠位相 (vague topology) を入れたものを配置空間という。

定義から s は

$$s = \sum_i \delta_{s_i} \quad (0.1)$$

という形をしている。ここで $\{s_i\}$ は集積点を持たない S の点列で、 δ_a は点 a に集中したデルタ測度である。 $\{s_i\}_i$ は S 内のラベルを付けない可算個の粒子系を表現する。ラベル無粒子系の表現は集合として表現するなど、異なる流儀もあるが測度として表現すると下に説明するように自然に位相が入るので便利である。

各 $A \in \mathcal{B}(S)$ に対して $s(A)$ は A 内の粒子の個数を表現し、非負整数および無限大の値をとる確率変数になる。仮定から定義から s はラドン測度だから、コンパクト集合に含まれる粒子数は常に有限になる。定義から、

$$\mathcal{S} = \{s; s = \sum_i \delta_{s_i}, s(K) < \infty \ (\forall \text{コンパクト } K \subset S)\} \quad (0.2)$$

^{r:11}注意 0.1. $s = \sum_{i=1}^{\infty} \delta_{\frac{1}{i}}$ とする。この時 s は $(0, \infty)$ 上の配置空間の元だが、 $[0, \infty)$ 上の配置空間の元ではない。実際、閉区間 $[0, 1]$ は $[0, \infty)$ のコンパクト集合で、かつ $s([0, 1]) = \infty$ となる。

S は Polish 空間となる。 S がコンパクトなら \mathcal{S} もコンパクトになる。

$S = \mathbb{R}^d$ の時、 S の部分集合 A が相対コンパクトである必要十分条件は、ある自然数列 $\{a_r\}$ が存在し $A \subset \{s \in \mathcal{S}; s(D_r) \leq a_r\} \ (\forall r \in \mathbb{N})$ となることである^{resnick}[62]。ただし $D_r = \{|x| \leq r\}$ 。

S の漠位相について、次の事実がよく知られている。

^{1:21}**補題 0.1.** $A \in \mathcal{B}(\mathcal{S})$ 、 A 上の配置空間を \mathcal{A} とする。漠位相のもとで次が成立する。

- (1) A が Polish 空間なら \mathcal{A} も Polish 空間である。
- (2) A が閉集合なら \mathcal{A} も閉集合である。
- (3) A がコンパクトなら \mathcal{A} もコンパクトである。

1

^{1:22}**補題 0.2.** $A \in \mathcal{B}(\mathcal{S})$ 、 A 上の配置空間を \mathcal{A} とする。 d を S の距離、 $s_0 \in S$ とし $B_r = \{d(x, s_0) < r\}$ とおく。 A が相対コンパクトである必要十分条件はある相対コンパクト集合の増大列 $\{K_r\}$ が存在し

$$A \subset \{s \in \mathcal{S}; s(B_r) \leq a_r \ (\forall r)\} \quad (0.3)$$

$(S, \mathcal{B}(S))$ 上の確率測度を点過程 (point process) あるいは確率点場 (random point field) といい。

¹ 混乱している。 \mathbb{R}^d ならいいが、ポーランド空間だとどうしよう

Poisson 点過程 m を $(S, \mathcal{B}(S))$ のラドン測度とする. S 上の点過程 μ は次の条件を満たす時, 強度測度 m の Poisson 点過程と呼ばれる.

$$\begin{aligned} s(A) \text{ は } \mu \text{ の下で平均 } m(A) \text{ の Poisson 分布に従う, ただし } m(A) < \infty & \quad \text{:poi1} \\ \{A_1, \dots, A_n\} \text{ が互いに素ならば, } \{s(A_1), \dots, s(A_n)\} \text{ は独立} & \quad \text{:poi2} \end{aligned} \quad (0.4)$$

特に $S = \mathbb{R}^d$ で強度測度 m が Lebesgue 測度の時, 付随する Poisson 点過程は様々な不変性を持ち, 配置空間上の Lebesgue 測度の役割を果たす.

周期的点過程 $S = \mathbb{R}^d$ する. d 個の独立なベクトル $\{v_1, \dots, v_d\}$ で張られる d 次元トーラス $\mathbf{T} = \{x \in \mathbb{R}^d; x = \sum_{i=1}^d \alpha_i v_i, 0 \leq \alpha_i < 1\}$ 上の一様分布を λ とする. 写像 $f: \mathbf{T} \rightarrow S$ を $f(x) = \sum_{z \in \mathbb{Z}^d} \delta_{x+z \cdot \mathbf{v}}$ で与える. ただし $\mathbf{v} = (v_1, \dots, v_d)$, \cdot は \mathbb{R}^d の標準内積である. このとき像測度 λ^f を周期的点過程という.

Lebesgue 測度を強度測度とする Poisson 点過程 Λ と周期的点過程はともに \mathbb{R}^d の配置空間上の平行移動不変な点過程だが, 性質は両極端である. Λ は最もランダムな点過程であるのに対して, 周期的点過程は, 本質的に d 次元の Lebesgue 測度のランダムネスしか持たない, 有限次元的な点過程である.

これら二つを両極端な場合として, その中間的な場合を考察していく.

相関関数・密度関数 m を S 上のラドン測度とする. 対称な局所可積分関数 $\rho^n: S^n \rightarrow [0, \infty)$ は (0.6) を満たす時, 点過程 μ の m に対する n 点相関関数 (n point correlation function) と呼ばれる.

$$\int_{A_1^{k_1} \times \dots \times A_j^{k_j}} \rho^n(x_1, \dots, x_n) \prod_{i=1}^n m(dx_i) = \int_S \prod_{i=1}^j \frac{s(A_i)!}{(s(A_i) - k_i)!} d\mu \quad \text{:28} \quad (0.6)$$

ここで $A_1, \dots, A_j \subset S$ は互いに素な可測集合, k_1, \dots, k_j は $k_1 + \dots + k_j = n$ を満たす自然数とする. $s(A_i) - k_i < 0$ の時は, $s(A_i)! / (s(A_i) - k_i)! = 0$ とみなす. 次の (0.7) を満たす時, $\{\rho^n\}_{n \in \mathbb{N}}$ を相関関数にもつ μ は一意である [28].

$$\sum_{n=0}^{\infty} \left\{ \frac{1}{n!} \int_{A^n} \rho^n(x_1, \dots, x_n) \prod_{i=1}^n m(dx_i) \right\}^{-\frac{1}{n}} = \infty \quad (\forall A: m(A) < \infty) \quad \text{:1enard} \quad (0.7)$$

$\mu(s(A) < \infty) = 1$ とする. 対称な非負関数 $\sigma_A^n: A^n \rightarrow [0, \infty)$ は (0.8) を満たす時, 点過程 μ の m に対する A 上の n 密度関数 (n density function) とよばれる.

$$\mu_A(B) = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\mathbf{u}_n^{-1}(B)} \sigma_A^n(\mathbf{x}_n) \prod_{i=1}^n m(dx_i) \quad \text{:28d} \quad (0.8)$$

ここで $\mu_A := \mu \circ \pi_A^{-1}$, ただし $A \in \mathcal{B}(S)$ に対して, $\pi_A: S \rightarrow S$ は

$$\pi_A(s) = s(\cdot \cap A)$$

を表す. 更に, B は A 上の配置空間の可測部分集合, $\mathbf{x}_n = (x_1, \dots, x_n)$, $\mathbf{u}_n: A^n \rightarrow S$ は $\mathbf{u}_n(\mathbf{x}_n) = \sum_{i=1}^n \delta_{x_i}$ である. 定義から

$$1 = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{A^n} \sigma_A^n(\mathbf{x}_n) \prod_{i=1}^n m(dx_i) \quad \text{:28e} \quad (0.9)$$

この正規化による密度関数は、Janossy 密度と呼ばれる。相関関数の定義から、

$$\rho^n(\mathbf{x}_n) = \sum_{i=0}^{\infty} \frac{1}{i!} \int_{A^i} \sigma_A^{n+i}(\mathbf{x}_{n+i}) \prod_{j=1}^i m(dx_{n+j}) \quad (\mathbf{x}_n \in A^n) \quad (0.10)$$

ある定数 $c_1 > 0$, $c_2 < 1$ に対し $\sup_{n \in \mathbb{N}} \sup_{A^n} \rho^n(\mathbf{x}_n) c_1^n n^{-c_2 n} < \infty$ を満たせば、逆に

$$\sigma_A^n(\mathbf{x}_n) = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} \int_{A^i} \rho^{n+i}(\mathbf{x}_{n+i}) \prod_{j=1}^i m(dx_{n+j}) \quad (0.11)$$

次の測度の弱収束条件^[49, Lem. 11.1] は、行列式測度に対して有効である。

補題 0.3. $S = \mathbb{R}$ の時、点過程 μ_n , μ の相関関数 $\{\rho_n^m\}$, $\{\rho^m\}$ が (0.12) (0.13) を満たせば $\{\mu_n\}$ は μ に弱収束する。ここで $c_1 > 0$, $c_2 < 1$ である。

$$\sup_n \sup_{n \in \mathbb{N}} \sup_{D_r^n} \rho_n^n(\mathbf{x}_n) c_1^n n^{-c_2 n} < \infty \quad (\forall r \in \mathbb{N}) \quad (0.12)$$

$$\lim_{n \rightarrow \infty} \rho_n^n(\mathbf{x}_n) = \rho^n(\mathbf{x}_n) \quad \text{for a.e. } \mathbf{x}_n, \forall n \in \mathbb{N} \quad (0.13)$$

相関関数と平均・分散 $\langle f, s \rangle = \int_S f ds = \sum_i f(s_i)$ と表す。ただし $s = \sum_i \delta_{s_i} \in S$. $f \in C_0(S)$ あるいは、より一般に積分が意味を持つ時、次が成立する。

$$E^\mu[\langle f, s \rangle] = \int_S f(x) \rho^1(x) m(dx) \quad (0.14)$$

$$E^\mu[|\langle f, s \rangle|^2] = \int_S |f(x)|^2 \rho^1(x) m(dx) + \int_{S^2} f(x) \bar{f}(y) \rho^2(x, y) m(dx) m(dy) \quad (0.15)$$

$$\text{Var}^\mu[\langle f, s \rangle] = \int_S |f(x)|^2 \rho^1(x) m(dx) + \int_{S^2} f(x) \bar{f}(y) \mathcal{T}(x, y) m(dx) m(dy) \quad (0.16)$$

ここで $\mathcal{T}(x, y) = \rho^2(x, y) - \rho^1(x) \rho^1(y)$ は 2cluster 関数と呼ばれる。^[37] と^[11] では定義の符号が違う。 μ が Hermite 対称な核 $K(x, y)$ から生成される行列式測度ならば、

$$\text{Var}^\mu[\langle f, s \rangle] = \int_S |f(x)|^2 K(x, x) m(dx) - \int_{S^2} f(x) \bar{f}(y) |K(x, y)|^2 m(dx) m(dy)$$

Palm 測度・Campbell 測度 $\mu_{\mathbf{x}}$ が $\mathbf{x} = (x_1, \dots, x_k)$ で条件付けた μ の Palm 測度とは、次で与えられる μ の条件付き確率のことをいう。

$$\mu_{\mathbf{x}} = \mu(\cdot - \sum_{i=1}^k \delta_{x_i} \mid s(\{x_i\}) \geq 1 \text{ for } i = 1, \dots, k). \quad (0.17)$$

より正確には、(0.17) は reduced Palm 測度とよばれる。つまり、条件づけた粒子の分を差っ引いた分を考えているからである。

μ^k が k -Campbell 測度とは、次式で定義される $S^k \times S$ 上の測度のことをいう。

$$\mu^k(A \times B) = \int_A \mu_{\mathbf{x}}(B) \rho^k(\mathbf{x}) \prod_{j=1}^k m(dx_j). \quad (0.18)$$

1 Gibbs 測度・行列式測度

ポーランド空間 S 内を運動する粒子が、自由ポテンシャル $\Phi : S \rightarrow \mathbb{R} \cup \{\infty\}$ で媒質 S と相互作用し、また干渉ポテンシャル $\Psi : S \times S \rightarrow \mathbb{R} \cup \{\infty\}$ で粒子間の相互作用をとする。 $\Psi(x, y) = \Psi(y, x)$ とする。この時、形式的な Hamiltonian \mathcal{H} は

$$\mathcal{H}(s) = \sum_i \Phi(s_i) + \sum_{i < j} \Psi(s_i, s_j) \quad (1.1) \quad :G1$$

で与えられ、対応する粒子系の平衡分布は、直感的には次式で記述される。

$$\mu(ds) = Z^{-1} e^{-\mathcal{H}(s)} \prod_i ds_i = Z^{-1} e^{-\sum_i \Phi(s_i) - \sum_{i < j} \Psi(s_i, s_j)} \prod_i ds_i \quad (1.2) \quad :G2$$

しかし無限体積の時、例えば $S = \mathbb{R}^d$ で $\int_S e^{-\Phi} ds = \infty$ の時、表示に Lebesgue 測度の無限直積を含むため、(1.2)はそのままでは正当化できない。そのためまず無限粒子系を個々の粒子を区別しない配置空間の元として捉える。以下、無限粒子系の平衡分布を構成する二つの方法を説明する。

Gibbs 測度 条件付き確率を用いて (1.2)を正当化する方法について述べる。以下、 $S = \mathbb{R}^d$, $D_r = \{|x| \leq r\}$ とする。 $\pi_r(s) = s(\cdot \cap D_r)$, $\pi_r^c(s) = s(\cdot \cap D_r^c)$ とおく。

$$\mathcal{H}_r(s) = \sum_{s_i \in D_r} \Phi(s_i) + \sum_{i < j, s_i, s_j \in D_r} \Psi(s_i, s_j) \quad (1.3) \quad :G9$$

$$\mathcal{H}_{r,\xi}(s) = \mathcal{H}_r(s) + \sum_{s_i \in D_r, \xi_j \in D_r^c} \Psi(s_i, \xi_j) \quad (\xi = \sum_i \delta_{\xi_i} \in S) \quad (1.4) \quad :G3$$

\mathbb{R}^d 上の点過程 μ が (Φ, Ψ) -カノニカル Gibbs 測度とは $\forall r, m \in \mathbb{N}$, μ -a.s. ξ に対し

$$\mu(\pi_r \in \cdot | \pi_r(D_r) = m, \pi_r^c(s) = \pi_r^c(\xi)) = Z^{-1} e^{-\mathcal{H}_{r,\xi}(s)} 1_{S_r^m}(s) \Lambda(ds) \quad (1.5) \quad :G4$$

ここで $S_r^m = \{s(D_r) = m\}$, Λ は Lebesgue 測度を強度とする Poisson 点過程である。 (1.5) の形の条件付き確率とポテンシャルの関係式を **DLR 方程式** (Dobrushin-Lanford-Ruelle equation) という。 Ψ は超安定かつ Ruelle の意味で正則の時、Ruelle クラスのポテンシャルという [63]¹⁾。 Ruelle クラスのポテンシャルを持つ Gibbs 測度は、Poisson 点過程に近い性質を持つクラスである。一方、 d 次元 Coulomb ポテンシャルは、 $n \geq d-2$ の時 \mathbb{R}^n において Ruelle の意味の正則性を満たさない。この時、 μ が平行移動不変な点過程ならば、 μ -a.s. ξ に対して (1.4)の最終項が発散し、DLR 方程式に意味がつかない。従って、カノニカル Gibbs 測度とならない。

行列式測度 m を S のラドン測度、 $K : S^2 \rightarrow \mathbb{C}$ とする。 m に対する相関関数が

$$\rho^n(\mathbf{x}_n) = \det[K(x_i, x_j)]_{i,j=1}^n \quad (1.6) \quad :det1$$

で与えられる点過程 μ を (K, m) で生成される行列式測度 (determinantal measure) という。行列式点過程、行列式確率点場とも呼ばれる。

定理 1.1 ([65, ?])^{1:det2}. (K, m) が (1.7)-(1.9)を満たす時、 (K, m) で生成される行列式測度 μ が一意に存在する。また (1.7) と (1.8)を満たす K が、行列式測度を生成するための必要十分条件

は (1.9) である.

$$K(x, y) = \overline{K(y, x)} \quad (K \text{ は Hermite 対称}) \quad \text{:det-a} \quad (1.7)$$

$$K \text{ は } L^2(S, m) \text{ 上, 局所 trace クラス作用素} \quad \text{:det-b} \quad (1.8)$$

$$0 \leq K \leq \text{Id} \quad (\text{つまり } \text{Spec}(K) \subset [0, 1]) \quad \text{:det-c} \quad (1.9)$$

ここで $Kf(x) = \int_S K(x, y)f(y)m(dy)$. (1.8) は K を任意のコンパクト集合 A に制限した時, K が $L^2(A, m)$ 上の *trace* クラスになることを意味する.

行列式測度は、様々な意味で非常にうまくできた対象である。点過程の世界で、まさにブラウン運動や Gauss 過程のやくわりを果たしている。とくに、行列式過程が核関数 $K(x, y)$ で本質的に規定されている点が Gauss 過程よく似ていることが、指摘されている。

次の定理は、無限次元確率微分方程式の解の一意性 ($\beta = 2$ かつクーロン interaction の場合) に本質的な役割を果たす。

^{1:tail}**定理 1.2.** 行列式過程は *tail sigma field* $\mathcal{T}(S)$ が自明になる。つまり

$$\mathcal{T}(S) = \bigcap_{r=1}^{\infty} \sigma[\pi_r^c]$$

とおくと

$$\mu(A) \in \{0, 1\} \quad \text{for all } A \in \mathcal{T}(S).$$

この結果の系として、一般に平行移動が定義された S 上の μ を考えると次が成立する。

^{1:erg}**定理 1.3.** 平行移動 (*shift*) 不変な行列式過程は、*shift* について *ergodic* である。

空間 S の可測部分集合 A 上の配置空間を S_A と表す。 S_A を自然に S の部分集合とみなす。行列式測度 μ の S_A への制限は、再び行列式測度になる。実際、 μ が (K, m) で生成される行列式測度の時、像測度 $\mu \circ \pi_A^{-1}$ は (K_A, m_A) で生成される (A 上の) 行列式測度となる。ただし $K_A(x, y) = 1_A(x)K(x, y)1_A(y)$, $m_A(dx) = 1_A(x)m(dx)$. また Palm 測度も行列式測度である。

^{1:det3}**定理 1.4** ([65]). ^{ST03} (K, m) が (1.7)–(1.9) を満たすとし、 μ を (K, m) で生成される行列式測度とする。この時、点 $a \in S$ で条件付けた Palm 測度 μ_a は (K_a, m) で生成される行列式測度になる。ただし K_a は次式で与えられる。

$$K_a(x, y) = K(x, y) - \frac{K(x, a)K(a, y)}{K(a, a)} \quad \text{:det-palm} \quad (1.10)$$

核関数の重要なクラスは \mathbb{R} の直交多項式で与えられる。 $w: \mathbb{R} \rightarrow [0, \infty)$ とする。 $\{p_m\}_{m=0}^{n-1}$ を *monic* (最高次の係数が 1) な多項式の列で p_m は m 次多項式、また測度 $w dx$ に対して直交するとする。この時

$$K^{(n)}(x, y) = \{w(x)w(y)\}^{\frac{1}{2}} \sum_{m=0}^{n-1} \frac{p_m(x)p_m(y)}{\int p_m(t)^2 w(t) dt} \quad \text{:poly} \quad (1.11)$$

とおくと、 $K^{(n)}$ は定理 ^{1:det2}1.1 の条件を満たし、 $(K^{(n)}, dx)$ で生成される行列式測度が存在する。次の和公式は $n \rightarrow \infty$ を計算する鍵となる [11]。

^{1:cd}
定理 1.5 (Christoffel-Darboux の公式).

$$K^{(n)}(x, y) = \frac{\{w(x)w(y)\}^{\frac{1}{2}}}{\int p_{n-1}(t)^2 w(t) dt} \cdot \frac{p_n(x)p_{n-1}(y) - p_{n-1}(x)p_n(y)}{x - y} \quad (1.12) \quad \text{:cd1}$$

(1.12)から直ちに対角成分の表示を得る.

$$K^{(n)}(x, x) = \frac{w(x)}{\int p_{n-1}(t)^2 w(t) dt} \cdot \{p'_n(x)p_{n-1}(x) - p'_{n-1}(x)p_n(x)\} \quad (1.13) \quad \text{:cd2}$$

点過程の対数微分・準 Gibbs 測度 無限体積 Gibbs 測度を Coulomb ポテンシャルに拡張するために 2 つの概念を導入する. 以下 $S = \mathbb{R}^d$, μ は S 上の点過程とする.

- $d^\mu = (d_m^\mu)_{m=1, \dots, d}$ が次の条件を満たす時, μ の対数微分という[48]:

$$d^\mu \in \{L_{\text{loc}}^1(\mu^1)\}^d \quad (1.14) \quad \text{:261}$$

$$\int_{S \times S} d^\mu f d\mu^1 = - \int_{S \times S} \nabla_x f d\mu^1 \quad (\forall f \in C_0^\infty(S) \otimes \mathcal{D}_0). \quad (1.15) \quad \text{:26m}$$

ただし $\nabla_x f = (\frac{\partial f(x, s)}{\partial x_m})_{m=1, \dots, d}$, μ^1 は (0.18) で定義された 1-Campbell 測度, \mathcal{D}_0 は S 上の局所的かつ滑らかな関数全体である. 対数微分は, 直感的には $d^\mu = \nabla_x \log \mu^1$ であり, 一つの粒子 x (tagged particle) が, 他の無限個の粒子 $s = \sum \delta_{s_i}$ からどのような力を受けるかを表現している. μ が (Φ, Ψ) -カノニカル Gibbs 測度ならば,

$$d^\mu(x, s) = -\nabla_x \Phi(x) - \sum_i \nabla_x \Psi(x, s_i) \quad (1.16) \quad \text{:26n}$$

DLR 方程式 (1.5) と異なり, この表記は Ψ が Coulomb ポテンシャルでも無限粒子系に対して意味を持ちうる. sine 点過程や Ginibre 点過程の場合を後で計算する.

- μ が次の条件を満たす時, (Φ, Ψ) -準 Gibbs 測度という[49]: ある自然数の増大列 $\{b_r\}$ と測度の列 $\{\mu_{r,k}^m\}$ が存在し, 各 $r, m \in \mathbb{N}$ に対し $\mu_{r,k}^m$ が次を満たす:

$$\mu_{r,k}^m \leq \mu_{r,k+1}^m \quad (\forall k), \quad \lim_{k \rightarrow \infty} \mu_{r,k}^m = \mu(\cdot \cap S_{b_r}^m) \quad \text{weakly}, \quad (1.17) \quad \text{:qg1}$$

更に $\forall r, m, k \in \mathbb{N}$ と $\mu_{r,k}^m$ -a.e. $s \in S$ に対して,

$$c_1^{-1} e^{-\mathcal{H}_{b_r}(x)} 1_{S_{b_r}^m}(x) \Lambda(dx) \leq \mu_{r,k,s}^m(dx) \leq c_1 e^{-\mathcal{H}_{b_r}(x)} 1_{S_{b_r}^m}(x) \Lambda(dx). \quad (1.18) \quad \text{:qg2}$$

ここで c_1 は $r, m, k, \pi_{\{|x| > b_r\}}(s)$ に依存する正定数, 更に

$$\mu_{r,k,s}^m(dx) = \mu_{r,k}^m(\pi_{\{|x| \leq b_r\}} \in dx | \pi_{\{|x| > b_r\}}(s)). \quad (1.19) \quad \text{:qg4}$$

Λ , $S_{b_r}^m$, $\mathcal{H}_{b_r}(x)$ は Gibbs 測度の節で定義されている.

カノニカル Gibbs 測度は準 Gibbs 測度である. Lebesgue 測度を基礎の測度にする行列式測度はすべて準 Gibbs 測度であると期待されるがまだ証明されていない. Yoo^[73]により核 K が $0 \leq K < 1$, つまり固有値 1 を持たない時, 行列式測度が Gibbs 測度であることが示されている. しかしランダム行列に関するものは無限体積において 1 を固有値として含む. 尚, sine 点過程や Ginibre 点過程については, 準 Gibbs 性が示されている.

点過程の対数微分・準 Gibbs 測度 2 : 確率力学への応用

定理 1.6 (^{1:ud}[49, 50]). μ が (Φ, Ψ) 準 Gibbs 測度であり, (Φ, Ψ) が上半連続ならば自然な *Dirichlet* 形式を用いて μ 可逆な拡散過程が構成できる.

さらに対数微分が存在すれば, それを用いて, 確率力学を無限次元確率微分方程式の解 (弱解) として表現できる.

定理 1.7 (^{1:isde}[48]). 定理 ^{1:ud}1.6 の仮定に加えて, 対数微分 d^μ が存在し, かつ, 非衝突および非爆発条件を満たすとす. すると, *unlabeled diffusion* X :

$$X_t = \sum_{i=1}^{\infty} \delta_{X_t^i}$$

の各成分は, (最初にラベルを付けたとして) つぎの無限次元確率微分方程式の解である.

$$dX_t^i = dB_t^i + \frac{1}{2} d^\mu(X_t^i, \sum_{j \neq i} \delta_{X_t^j}) dt \quad (i \in \mathbb{N})$$

定理 1.8 (^{1:isde}[57]). 定理 ^{1:isde}1.8 の仮定に加えて, μ が *tail* 自明ならば, 上の無限次元確率微分方程式は一意的強解をもつ.

定理 1.9 (^{1:isde2}[57]). 全ての準 Gibbs 測度は, *tail* 自明な成分に分解できる. 上の無限次元確率微分方程式は, その成分を不変にする一意的強解をもつ.

これらの定理に現れる条件はすべて点過程 μ の言葉で直接的に表せる, という意味で幾何的である.

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Infinite-dimensional stochastic differential equations arising from random matrix theory

2014/9/1/Mon–2014/9/5/Wed Warwick

UK-Japan Stochastic Analysis School (JSPS Core-to-Core programme)

Total plan:

- 1'st Talk: Examples & strategy of the proof
- 2'nd Talk: Weak solutions
- 3'rd Talk: Strong solutions and pathwise uniqueness

Outline of 1'st talk:

- Dynamical soft edge scaling limit: Airy_β RPFs ($\beta = 1, 2, 4$)
- Dynamical bulk scaling limit: Sine RPFs and an SDE gap
- Ginibre and Bessel RPFs
- Historical back ground of interacting Brownian motions
- Strategy: Outline of the proof.

Geometric scaling limit

Geometric soft edge/bulk scaling limits of Gaussian ensembles

- GUE (Gaussian unitary ensemble): Gaussian random matrices:

$$M = \begin{pmatrix} M_{11} & M_{12} & \cdots & \cdots & M_{1N} \\ M_{21} & M_{22} & \cdots & \cdots & M_{2N} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ M_{N1} & M_{N2} & \cdots & \cdots & M_{NN} \end{pmatrix} \quad (1)$$

such that M are $N \times N$ -hermitian matrices whose entries satisfying that

$$M_{ij} = \frac{G_{ij,1} + \sqrt{-1}G_{ij,2}}{\sqrt{2}} \quad (i < j), \quad M_{kk} = G_k \quad (2)$$

and that $G_{ij,1}$, $G_{ij,2}$, G_k are i.i.d. Gaussian random variables with mean 0 and variance 1.

- We define GOE and GSE similarly as real/quaternion symmetric Gaussian random variables.

Geometric scaling limit

- The distribution of eigen values of the G(O/U/S)E Random Matrices are given by ($\beta = 1, 2, 4$)

$$m_{\beta}^N(d\mathbf{x}_N) = \frac{1}{Z} \prod_{i < j}^N |x_i - x_j|^{\beta} e^{-\frac{\beta}{4} \sum_{i=1}^N |x_i|^2} d\mathbf{x}_N, \quad (3)$$

This means the system can be regarded as particles interacting through logarithmic potential (2D Coulomb potentials).

- Wigner's theorem: The distribution of

$$\frac{1}{N} \sum_{i=1}^N \delta_{x_i/\sqrt{N}} \quad \text{under } m_{\beta}^N$$

converges to the semi-circle law

$$\varsigma(x)dx = \frac{1}{2\pi} \sqrt{4 - x^2} dx \quad (4)$$

- This convergence corresponds to the law of large numbers.

Bulk/Soft edge scaling

$$m_{\beta}^N(d\mathbf{x}_N) = \frac{1}{Z} \prod_{i < j}^N |x_i - x_j|^{\beta} e^{-\frac{\beta}{4} \sum_{i=1}^N |x_i|^2} d\mathbf{x}_N, \quad (3)$$

$$\varsigma(x)dx = \frac{1}{2\pi} \sqrt{4 - x^2} dx$$

- **Bulk scaling:** For $-2 < \theta < 2$ take $x_i = (s_i - \theta)/\sqrt{N}$ in (3):

$$\mu_{\text{sin},\beta,\theta}^N(ds_N) = \frac{1}{Z} \prod_{i < j}^N |s_i - s_j|^{\beta} \prod_{k=1}^N e^{-\beta |s_k - \theta|^2 / 4N} ds_N \quad (5)$$

- **Soft edge scaling:** Take $x_i \mapsto 2\sqrt{N} + s_i N^{-1/6}$ in (3):

$$\mu_{\text{Ai},\beta}^N(ds_N) = \frac{1}{Z} \prod_{i < j}^N |s_i - s_j|^{\beta} e^{-\frac{\beta}{4} \sum_{i=1}^N |2\sqrt{N} + N^{-1/6} s_i|^2} ds_N.$$

Airy RPF – Soft edge scaling limit

Soft edge scaling limit

Airy RPF: $\mu_{\text{Ai},\beta}$ ($\beta = 1, 2, 4$)

- Take the scaling $x_i \mapsto 2\sqrt{N} + s_i N^{-1/6}$ in

$$m_{\beta}^N(d\mathbf{x}_N) = \frac{1}{Z} \prod_{i < j}^N |x_i - x_j|^{\beta} e^{-\frac{\beta}{4} \sum_{i=1}^N |x_i|^2} d\mathbf{x}_N$$

and set

$$\mu_{\text{Ai},\beta}^N(ds_N) = \frac{1}{Z} \prod_{i < j}^N |s_i - s_j|^{\beta} e^{-\frac{\beta}{4} \sum_{i=1}^N |2\sqrt{N} + N^{-1/6} s_i|^2} ds_N.$$

Then $\mu_{\text{Ai},\beta}^N$ converge to Airy RPF $\mu_{\text{Ai},\beta}$:

$$\lim_{N \rightarrow \infty} \mu_{\text{Ai},\beta}^N = \mu_{\text{Ai},\beta}$$

Airy RPF – Soft edge scaling limit

- $\beta = 2 \Rightarrow \mu_{\text{Ai},\beta}$ is a determinantal RPF given by (K_{Ai}, dx) :

$$K_{\text{Ai}}(x, y) = \frac{\text{Ai}(x)\text{Ai}'(y) - \text{Ai}'(x)\text{Ai}(y)}{x - y}$$

Here $\text{Ai}(\cdot)$ is the Airy function.

The correlation function ρ_{Ai}^n is defined as

$$\rho_{\text{Ai}}^n(\mathbf{x}) = \det[K_{\text{Ai}}(x_i, x_j)]_{i,j=1}^n.$$

- If $\beta = 1, 4$, the correlation func of $\mu_{\text{Ai},\beta}$ are given by similar formula of quaternion determinant.
- We discuss a dynamical counter part of this scaling limit.

Airy RPF – Soft edge scaling limit

- I give here minimal definition.
- Let $S = \mathbb{R}^d, [0, \infty),$ e.t.c.. S : configuration space over S
$$S = \{s = \sum_i \delta_{s_i}; s_i \in S, s(|s| < r) < \infty (\forall r \in \mathbb{N})\}$$
- S is a Polish space with the vague topology.
- A prob meas. μ on S is called a random point field (RPF) on S .
- S is the set of **unlabeled** particles.
- $S^{\mathbb{N}}$ is the space of **labeled** particles.
- A symmetric function ρ^n is called the n -correlation function of μ w.r.t. Radon m. m if

$$\int_{A_1^{k_1} \times \dots \times A_m^{k_m}} \rho^n(\mathbf{x}_n) \prod_{i=1}^n m(dx_i) = \int_S \prod_{i=1}^m \frac{s(A_i)!}{(s(A_i) - k_i)!} d\mu$$

for any disjoint $A_i \in \mathcal{B}(S), k_i \in \mathbb{N}$ s.t. $k_1 + \dots + k_m = n$.

Airy RPF – Soft edge scaling limit

- μ is called the **determinantal RPF** generated by (K, m) if its n -correlation functions ρ^n is given by

$$\rho^n(\mathbf{x}_n) = \det[K(x_i, x_j)]_{1 \leq i, j \leq n}$$

- It is known that (K, m) determines the RPF uniquely.
- The N -particle system of Airy RPF is a determinantal RPF whose kernel $K_{\text{Ai}}^N(x, y)$ is given by orthogonal polynomials.

- The convergence of $\mu_{\text{Ai}, \beta}^N$ follows from that of correlation functions.
- This follows from that of kernels $K_{\text{Ai}}^N(x, y)$.
- This follows from a calculation of orthogonal polynomials (special functions).

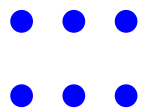
Airy RPF – Dynamical soft edge scaling limit

- We return to a dynamical soft edge scaling limit.
- From

$$\mu_{\text{Ai},\beta}^N(ds_N) = \frac{1}{Z} \prod_{i < j} |s_i - s_j|^\beta e^{-\frac{\beta}{4} \sum_{i=1}^N |2\sqrt{N} + N^{-1/6} s_i|^2} ds_N$$

we deduce the SDE of the N particle system:

$$dX_t^{N,i} = dB_t^i + \frac{\beta}{2} \sum_{j=1, j \neq i}^N \frac{1}{X_t^{N,i} - X_t^{N,j}} dt - \frac{\beta}{2} \left\{ N^{1/3} + \frac{1}{2N^{1/3}} X_t^{N,i} \right\} dt$$



- From

$$\mu_{\text{Ai},\beta}^N(ds_N) = \frac{1}{Z} \prod_{i<j} |s_i - s_j|^\beta e^{-\frac{\beta}{4} \sum_{i=1}^N |2\sqrt{N} + N^{-1/6} s_i|^2} ds_N$$

we deduce the SDE of the N particle system:

$$dX_t^{N,i} = dB_t^i + \frac{\beta}{2} \sum_{j=1, j \neq i}^N \frac{1}{X_t^{N,i} - X_t^{N,j}} dt - \frac{\beta}{2} \left\{ N^{1/3} + \frac{1}{2N^{1/3}} X_t^{N,i} \right\} dt$$

- Indeed, $X_t^{N,i}$ are associated with the Dirichlet form:

$$\mathcal{E}^{\mu_{\text{Ai},\beta}^N}(f, g) = \int_{\mathbb{R}^N} \frac{1}{2} \sum_i^N \frac{\partial f}{\partial s_i} \frac{\partial g}{\partial s_i} \mu_{\text{Ai},\beta}^N(ds_N) \text{ on } L^2(\mathbb{R}^N, \mu_{\text{Ai},\beta}^N).$$

Then, by integration by parts, the generator is

$$-L^N = \frac{1}{2} \Delta_N + \frac{\beta}{2} \sum_{i=1}^N \left[\sum_{j \neq i}^N \frac{1}{s_i - s_j} - \frac{\beta}{2} \left\{ N^{1/3} + \frac{s_i}{2N^{1/3}} \right\} \right] \frac{\partial}{\partial s_i}$$

Airy RPF – Dynamical soft edge scaling limit

- The SDE of the N particle system:

$$dX_t^{N,i} = dB_t^i + \frac{\beta}{2} \sum_{j=1, j \neq i}^N \frac{1}{X_t^{N,i} - X_t^{N,j}} dt - \frac{\beta}{2} \left\{ N^{1/3} + \frac{1}{2N^{1/3}} X_t^{N,i} \right\} dt$$

- The dynamics are also given by the space-time correlation functions.
- **Problem:** What SDE does the limit $\mathbf{X}_t = \lim_{N \rightarrow \infty} \mathbf{X}_t^N$ satisfy?

Does $\lim_{N \rightarrow \infty} \left\{ \sum_{j=1, j \neq i}^N \frac{1}{X_t^{N,i} - X_t^{N,j}} - N^{1/3} \right\}$ converge ?

How to solve the limit ISDE?

Airy RPF – Dynamical soft edge scaling limit

For a configuration $s = \sum_i \delta_{s_i}$, let $\ell(s) = (s_1, s_2, \dots) = s \in \mathbb{R}^{\mathbb{N}}$ be a label such that $s_1 > s_2 > \dots$, which is well defined for $\mu_{\text{Ai},\beta}^\ell$ -a.s..

Thm 1 (O.-Tanemura '14). **[Existence of strong solutions]**

Let $\beta = 1, 2, 4$. Define ISDE (6) of $\mathbf{X} = (X^i)_{i \in \mathbb{N}}$ as

$$dX_t^i = dB_t^i + \frac{\beta}{2} \lim_{r \rightarrow \infty} \left\{ \left(\sum_{j \neq i, |X_t^j| < r} \frac{1}{X_t^i - X_t^j} \right) - \int_{|x| < r} \frac{\varrho(x)}{-x} dx \right\} dt \quad (6)$$

$$\varrho(x) = \frac{\sqrt{-x}}{\pi} \mathbf{1}_{(-\infty, 0]}(x).$$

- For $\mu_{\text{Ai},\beta}^\ell$ -a.s.s, ISDE (6) has a strong solution with $\mathbf{X}_0 = s$.
- The associated unlabeled dynamics $X_t = \sum_{i=1}^{\infty} \delta_{X_t^i}$ is $\mu_{\text{Ai},\beta}$ -reversible.
- If $\beta = 2$ and $\mathbf{X}_0 \sim \mu_{\text{Ai},2}^\ell$, then $X_t^1 \sim F_2$. Here F_2 is the Tracy-Widom distribution and X_t^1 is the Airy process $\mathcal{A}(t)$.

Remarks:

- The key idea to derive the limit ISDE is to take the **rescaled** semi-circle law ς^N :

$$\begin{aligned}\varsigma^N(x) &:= N^{1/3} \varsigma\left(\frac{x}{N^{2/3}} + 2\right) \\ &= \frac{1_{(-4N^{2/3}, 0)}}{\pi} \sqrt{-x\left(1 + \frac{x}{4N^{2/3}}\right)}\end{aligned}$$

as the first approximation of the 1-correlation fun $\rho_{\text{Ai}, \beta}^{N, 1}$.

- We expect that our method can be applied to other soft edge scaling.
- The SDE gives a kind of Girsanov formula.

Airy RPF – Dynamical soft edge scaling limit

Thm 2 (O.-Tanemura '14). [Pathwise uniqueness]

Let $\beta = 1, 2, 4$. Then:

- Solutions of ISDE (6) of $\mathbf{X} = (X^i)_{i \in \mathbb{N}}$ starting at s

$$dX_t^i = dB_t^i + \frac{\beta}{2} \lim_{r \rightarrow \infty} \left\{ \left(\sum_{j \neq i, |X_t^j| < r} \frac{1}{X_t^i - X_t^j} \right) - \int_{|x| < r} \frac{\varrho(x)}{-x} dx \right\} dt \quad (6)$$

satisfying abs cont cond (7) are pathwise unique for $\mu_{\text{Ai}, \beta}^\ell$ -a.s.s.

$$\mu_{\text{Ai}, \beta, t} \circ X_t^{-1} \prec \mu_{\text{Ai}, \beta, t} \quad \text{for } \mu_{\text{Ai}, \beta}\text{-a.s. t.} \quad (7)$$

Here $\mu_{\text{Ai}, \beta, t}$ is a regular conditional probability w.r.t. to the tail σ -field \mathcal{T} of the configuration space. Namely

$$\mu_{\text{Ai}, \beta, t} = \mu_{\text{Ai}, \beta}(\cdot | \mathcal{T})(t), \quad \mathcal{T} = \bigcap_{r=1}^{\infty} \sigma[\pi_{S_r^c}]$$

where $\pi_A(s) = s(\cdot \cap A)$ is a projection on configuration space, and $S_r = \{|x| < r\}$. • • •

Airy RPF – Dynamical soft edge scaling limit

Thm 3 (O.-Tanemura '14). **[Pathwise uniqueness]**

Let $\beta = 1, 2, 4$. Then:

- Solutions of ISDE (6) of $\mathbf{X} = (X^i)_{i \in \mathbb{N}}$ starting at s

$$dX_t^i = dB_t^i + \frac{\beta}{2} \lim_{r \rightarrow \infty} \left\{ \left(\sum_{j \neq i, |X_t^j| < r} \frac{1}{X_t^i - X_t^j} \right) - \int_{|x| < r} \frac{\varrho(x)}{-x} dx \right\} dt \quad (6)$$

satisfying abs cont cond (7) are pathwise unique for $\mu_{\text{Ai},\beta}^\ell$ -a.s.s.

$$\mu_{\text{Ai},\beta,t} \circ X_t^{-1} \prec \mu_{\text{Ai},\beta,t} \quad \text{for } \mu_{\text{Ai},\beta}\text{-a.s. t.} \quad ((7))$$

Here $\mu_{\text{Ai},\beta,t}$ is a regular conditional probability w.r.t. to the tail σ -field \mathcal{T} of the configuration space.

- If $\beta = 2$, then \mathcal{T} is $\mu_{\text{Ai},\beta}$ -trivial. Hence the uniqueness holds.
- The solutions in Thm 1 satisfy (7). Hence tail preserving solutions exist uniquely.
- Weak solutions satisfying (7) are automatically unique strong solutions.

Airy RPF – Dynamical soft edge scaling limit : algebraic construction

If $\beta = 2$, then Johansson, Spohn, Katori-Tanemura & others show that there exist stochastic dynamics Z_t associated with $\mu_{\text{Ai},2}$ given by the space-time correlation function.

• Space-time correlation functions are given by the extended Airy kernel:

$$K_{\text{Ai}}(s, x; t, y) = \begin{cases} \int_0^\infty du e^{-u(t-s)/2} \text{Ai}(u+x) \text{Ai}(u+y), & t \geq s \\ -\int_{-\infty}^0 du e^{-u(t-s)/2} \text{Ai}(u+x) \text{Ai}(u+y), & t < s \end{cases}$$

The unlabeled process $Z_t = \sum_{i=1}^\infty \delta_{Z_t^i}$ is given by its moment generating function ($\mathbf{f} = (f_1, \dots, f_M)$, $\mathbf{t} = (t_1, \dots, t_M)$, $t_i < t_{i+1}$)

$$\Psi^{\mathbf{t}}[\mathbf{f}] = E[\exp\{\sum_{m=1}^M \int_{\mathbb{R}} f_m(x) Z_{t_m}(dx)\}]$$

defined as a Fredholm determinant

$$\Psi^{\mathbf{t}}[\mathbf{f}] = \text{Det}_{(s,t) \in I^2, (x,y) \in \mathbb{R}^2} [\delta_{st} \delta(x-y) + K_{\text{Ai}}(s, x; t, y) \chi_t(y)].$$

Here $I = \{t_1, \dots, t_M\}$ and $\chi_{t_m}(y) = e^{f_m(y)} - 1$,

Airy RPF – Dynamical soft edge scaling limit : algebraic construction

Thm 3 [O.-Tanemura, '14]

Let $\beta = 2$. Then these two dynamics are the same.

- This comes from the uniqueness of Dirichlet forms associated with these dynamics. To prove the uniqueness of Dirichlet forms, we use the uniqueness of weak solutions of the ISDE (6) .
- The first approach (ISDE) provides qualitative information, say, semimartingale property of each tagged particle, Hölder continuity of sample paths, non-collision property of tagged particles, and so on.
- The second construction gives quantitative information.
- By construction, if the total system start from the Airy₂ RPF $\mu_{\text{Ai},2}$, then the distribution of the top particle X_t^1 equals $F_{2,\text{edge}}(x)$, the 2 Tracy-Widom distribution.
- If we label the particles decreasing order as $X_t^i > X_t^{i+1}$, then the top particle X_t^1 is the Airy process $\mathcal{A}(t)$ studied by Spohn.

Airy RPF – Dynamical soft edge scaling limit

Let $\mathbf{X}_t^N = (X_t^{N,i})_{i=1}^N$ be the N -particle system as before:

$$dX_t^{N,i} = dB_t^i + \frac{\beta}{2} \sum_{j=1, j \neq i}^N \frac{1}{X_t^{N,i} - X_t^{N,j}} dt - \frac{\beta}{2} \left\{ N^{1/3} + \frac{1}{2N^{1/3}} X_t^{N,i} \right\} dt$$

Set $\mathbf{X}^{N,m}$ be the first m -component.

$$\mathbf{X}^{N,m} = (X_t^{N,1}, \dots, X_t^{N,m})$$

Thm 4 [O.-Tanemura, O.-Kawamoto] (**Finite-particle approximation**)

Let $\beta = 1, 2, 4$. Then for each $0 \leq \varphi \in L^1(\mu_{\text{Ai},\beta}^\ell)$ with $\int \varphi \mu_{\text{Ai},\beta}^\ell = 1$, $\mathbf{X}^{N,m}$ with $\mathbf{X}_0^{N,m} \sim \varphi \mu_{\text{Ai},\beta}^\ell$ converge to the first m -component \mathbf{X}^m of the solution of the limit ISDE weakly in $C([0, \infty); \mathbb{R}^m)$.

- When $\beta = 2$, we have two proofs.

Bulk scaling

Bulk scaling limit & an SDE gap

• • •

Bulk scaling limit & an SDE gap

- Bulk scaling:

For $-2 < \theta < 2$ take $x_i = (s_i - \theta)/\sqrt{N}$ in (3):

$$\mu_{\text{sin},\beta,\theta}^N(ds_N) = \frac{1}{Z} \prod_{i<j}^N |s_i - s_j|^\beta \prod_{k=1}^N e^{-\beta|s_k - \theta|^2/4N} ds_N \quad (8)$$

As $N \rightarrow \infty$, $\mu_{\text{sin},\beta,\theta}^N$ converge to the sine $_\beta$ RPF such that

$$\lim_{N \rightarrow \infty} \mu_{\text{Sine},\beta,\theta}^N = \mu_{\text{Sine},\beta,\theta}.$$

The right-hand side is independent of θ up to constant scaling.

If $\beta = 2$, then $\mu_{\text{Sine},2,\theta}$ is determinantal with kernel

$$K(x, y) = \sqrt{1 - \left(\frac{\theta}{2}\right)^2} \frac{\sin(x - y)}{\pi(x - y)}$$

- We next consider the dynamical counter part of this scaling limit.

Bulk scaling (dynamical)

$$\mu_{\sin, \beta, \theta}^N(ds_N) = \frac{1}{Z} \prod_{i < j}^N |s_i - s_j|^\beta \prod_{k=1}^N e^{-\beta |s_k - \theta|^2 / 4N} ds_N \quad (8)$$

- The associated N particle system is given by the SDE:

$$dX_t^i = dB_t^i + \frac{\beta}{2} \sum_{j \neq i}^N \frac{1}{X_t^i - X_t^j} dt - \frac{\beta}{4N} X_t^i dt + \frac{\beta\theta}{4} dt \quad (9)$$

- Very loosely, the associated ∞ particle system is given by

$$dX_t^i = dB_t^i + \frac{\beta}{2} \sum_{j \neq i}^{\infty} \frac{1}{X_t^i - X_t^j} dt + \frac{\beta\theta}{4} dt \quad (i \in \mathbb{N}).$$

This is not the case for $\theta \neq 0$.

Sine RPF – Limit ISDE

For a configuration $s = \sum_i \delta_{s_i}$, let $\ell(s) = (s_1, s_2, \dots) = s \in \mathbb{R}^{\mathbb{N}}$ be a label which is defined for $\mu_{\text{Sine},\beta}^\ell$ -a.s..

Limit ISDE:

Thm 5 [O.-Tanemura '14, O.-Kawamoto '14] **[Existence of strong solutions]**

Let $\beta = 1, 2, 4$. Define ISDE (6) of $\mathbf{X} = (X^i)_{i \in \mathbb{N}}$ as

$$dX_t^i = dB_t^i + \frac{\beta}{2} \lim_{r \rightarrow \infty} \left\{ \sum_{j \neq i, |X_t^j| < r} \frac{1}{X_t^i - X_t^j} \right\} dt \quad (10)$$

- For $\mu_{\text{Sine},\beta}^\ell$ -a.s.s, ISDE (10) has a strong solution with $\mathbf{X}_0 = s$.
- The associated unlabeled dynamics $X_t = \sum_{i=1}^{\infty} \delta_{X_t^i}$ is $\mu_{\text{Sine},\beta}$ -reversible.

Sine RPF - Dynamical bulk scaling limit

Thm 6 [O.-Tanemura '14] [Pathwise uniqueness]

Let $\beta = 1, 2, 4$. Then:

- Solutions of ISDE (10) of $\mathbf{X} = (X^i)_{i \in \mathbb{N}}$ starting at \mathbf{s}

$$dX_t^i = dB_t^i + \frac{\beta}{2} \lim_{r \rightarrow \infty} \left\{ \sum_{j \neq i, |X_t^j| < r} \frac{1}{X_t^i - X_t^j} \right\} dt \quad (10)$$

satisfying abs cont cond (11) are pathwise unique for $\mu_{\text{Sine}, \beta}^\ell$ -a.s. s.

$$\mu_{\text{Sine}, \beta, t} \circ X_t^{-1} \prec \mu_{\text{Sine}, \beta, t} \quad \text{for } \mu_{\text{Sine}, \beta}\text{-a.s. } t. \quad (11)$$

- If $\beta = 2$, then \mathcal{T} is $\mu_{\text{Sine}, \beta}$ -trivial. Hence the uniqueness holds.
- The solutions in Thm satisfy (11). Hence tail preserving solutions exist uniquely.
- Weak solutions satisfying (11) are automatically unique strong solutions.
- If $\beta = 2$, the solution equal to the stochastic dynamics given by space-time correlation functions (extended Sine kernels).

Sine RPF - Dynamical bulk scaling limit

Let $\mathbf{X}_t^N = (X_t^{N,i})_{i=1}^N$ be the N -particle system as before:

$$dX_t^{N,i} = dB_t^i + \frac{\beta}{2} \sum_{j \neq i}^N \frac{1}{X_t^{N,i} - X_t^{N,j}} dt - \frac{\beta}{4N} X_t^{N,i} dt + \frac{\beta\theta}{4} dt \quad (9)$$

Thm 7 [O.-Tanemura, O.-Kawamoto] (**Finite-particle approxim**)

Let $\beta = 1, 2, 4$. Then for each $0 \leq \varphi \in L^1(\mu_{\text{Ai},\beta}^\ell)$ with $\int \varphi \mu_{\text{Ai},\beta}^\ell = 1$, $\mathbf{X}^{N,m}$ with $\mathbf{X}_0^{N,m} \sim \varphi \mu_{\text{Ai},\beta}^\ell$ converge to the first m -component \mathbf{X}^m of the solution of the limit ISDE

$$dX_t^i = dB_t^i + \frac{\beta}{2} \lim_{r \rightarrow \infty} \left\{ \sum_{j \neq i, |X_t^j| < r} \frac{1}{X_t^i - X_t^j} \right\} dt \quad (10)$$

weakly in $C([0, \infty); \mathbb{R}^m)$.

- The limit ISDE (10) is independent of θ .
- In this sense, an SDE gap occurs.

Bessel RPF: hard edge scaling

Bessel RPF & a hard edge scaling

- Bessel RPFs $\mu_{\text{bes},2}^\alpha$ ($-1 < \alpha < \infty$) are probability measures on the configuration space S over $S = [0, \infty)$, whose n -point correlation functions ρ^n with respect to the Lebesgue measure are given by

$$\rho^n(x_1, \dots, x_n) = \det[K(x_i, x_j)]_{1 \leq i, j \leq n}. \quad (12)$$

Here $K(x, y)$ is called the Bessel kernel defined with the Bessel function J_α of order α such that for $x \neq y$

$$K(x, y) = \frac{J_\alpha(\sqrt{x})\sqrt{y}J'_\alpha(\sqrt{y}) - \sqrt{x}J'_\alpha(\sqrt{x})\sqrt{y}J_\alpha(\sqrt{y})}{2(x - y)}. \quad (13)$$

We note that $0 \leq K \leq \text{Id}$ as an operator on $L^2(S, dx)$.

- By definition $\mu_{\text{bes},2}^\alpha$ are determinantal random point fields with Bessel kernels K

Bessel RPF: hard edge scaling

Thm 8 [O.-Honda, '14] Let $\alpha > 1$ and $\beta = 2$. Let $\mu_{\text{bes},2}^\alpha$ be the Bessel $_2^\alpha$ RPF. Then the associated ISDE is given by

$$dX_t^i = dB_t^i + \frac{\alpha}{2X_t^i}dt + \frac{\beta}{2} \sum_{\substack{|X_t^j| < r \\ j \neq i}} \frac{1}{X_t^i - X_t^j} dt.$$

These ISDEs have unique, strong solutions (as in the same meaning of the previous theorems).

Bessel RPF: hard edge scaling

- These random point fields arise as a scaling limit at the hard left edge of the distributions $\mu_{\text{bes},2}^{\alpha,N}$ of the spectrum of the Laguerre ensemble.
- The random point fields $\mu_{\text{bes},2}^{\alpha}$ represent the thermodynamic limit of the N -particle systems $\mu_{\text{bes},2}^{\alpha,N}$, whose labeled densities $\sigma_{\alpha}^N(\mathbf{x})d\mathbf{x}$ are given by

$$\sigma_{\alpha}^N(\mathbf{x}) = \frac{1}{\mathcal{Z}_{\alpha}^N} e^{-\sum_{i=1}^N x_i/4N} \prod_{j=1}^N x_j^{\alpha} \prod_{k<l}^N |x_k - x_l|^2. \quad (14)$$

Thm 9 [O. Kawamoto] The associated N -particle system $\mathbf{X}_t^N = (X_t^{N,1}, \dots, X_t^{N,N})$ converge to the limit $\mathbf{X}_t = (X_t^i)_{i \in \mathbb{N}}$ in the same sense as before.

- $\beta = 1, 4$ is in progress.

Ginibre RPF

Ginibre RPF : Non-hermitian Gaussian random matrixes

- Ginibre RPF μ_{gin} is a determinantal RPF on \mathbb{C} (\mathbb{R}^2) with (K, g) .
- $g(z) = (1/\pi)e^{-|z|^2}$ is a Gauss measure on \mathbb{C} .
- K is an exponential kernel

$$K(x, y) = e^{x\bar{y}}.$$

- n -correlation function ρ^n of Ginibre RPF w.r.t. $g^n dx^n$ is defined as

$$\rho^n(x_1, \dots, x_n) = \det[K(x_i, x_j)]_{i,j=1}^n$$

- The N -particle system is given by

$$\mu^N(d\mathbf{x}_N) = \frac{1}{\mathcal{Z}} \prod_{i < j}^N |x_i - x_j|^2 g^n(\mathbf{x}_N) d\mathbf{x}_N$$

- μ^N is a determinantal RPF with (K^N, dx) such that

$$K^N(x, y) = \sum_{m=0}^{N-1} \frac{(x\bar{y})^m}{m!}$$

Ginibre RPF

Thm 11 [O., '13, O.-Tanemura '14]

Let μ_{gin} be a Ginibre RPF. Then the associated ISDE is given by the following, and has a unique strong solution as in the same meaning of the previous theorems.

$$dX_t^i = dB_t^i + \lim_{r \rightarrow \infty} \sum_{\substack{|X_t^i - X_t^j| < r \\ j \neq i}} \frac{X_t^i - X_t^j}{|X_t^i - X_t^j|^2} dt \quad (i \in \mathbb{N}).$$

The solution also satisfy the following ISDEs for all $a \in \mathbb{C}$:

$$dX_t^i = dB_t^i - (X_t^i - a)dt + \lim_{r \rightarrow \infty} \sum_{\substack{|a - X_t^j| < r \\ j \neq i}} \frac{X_t^i - X_t^j}{|X_t^i - X_t^j|^2} dt \quad (i \in \mathbb{N})$$

Ginibre RPF

The associated N -particle system is given by

$$dX_t^{N,i} = dB_t^i - X_t^{N,i} dt + \sum_{j \neq i}^N \frac{X_t^{N,i} - X_t^{N,j}}{|X_t^{N,i} - X_t^{N,j}|^2} dt.$$

Thm 12 The N -particle system

$$\mathbf{X}^N = (X_t^{N,1}, \dots, X_t^{N,N})$$

converge to the limit \mathbf{X} in the same sense as before.

Simulation!!

A historical background of IBMs

- Interacting Brownian motions in infinite-dimensions $\mathbf{X} = (X^i)_{i \in \mathbb{N}}$ are stochastic dynamics in $(\mathbb{R}^d)^{\mathbb{N}}$ given by ISDE

$$dX_t^i = dB_t^i - \frac{\beta}{2} \sum_{j \in \mathbb{N}, j \neq i} \nabla \Psi(X_t^i - X_t^j) dt \quad (i \in \mathbb{N})$$

Here Ψ is an interaction potential and β is inverse temperature. This ISDE has been studied by Lang ('79), Fritz ('87), Tanemura ('96), and others.

They construct **strong** solutions.

- So far Ψ is taken to be $C_0^3(\mathbb{R}^d)$ or exponential decay at infinity.
- Itô scheme (Picard approximation) is used here.

Known results.

$$dX_t^i = dB_t^i - \frac{\beta}{2} \sum_{j \in \mathbb{N}, j \neq i} \nabla \Psi(X_t^i - X_t^j) dt \quad (i \in \mathbb{N})$$

- There many interesting potentials Ψ with polynomial decay or unbounded at infinity:
- These are excluded by the classical approach based on Itô scheme.
- In this talk, we present a new scheme applicable to polynomial decay or logarithmic potentials:

$$\Psi(x) = -\log |x|.$$

This appears in random matrix theory and vortex dynamics. If $d = 1$, $\beta = 2$, and Ψ is as above, then the ISDE is

$$dX_t^i = dB_t^i + \sum_{j \neq i} \frac{1}{X_t^i - X_t^j} dt \quad (i \in \mathbb{N}).$$

Known results.

- Itô scheme uses Lipschitz continuity of coefficients, which does not hold in infinite dimensions.
- We localize ISDE with increasing sets H_k and exit times τ_{H_k} such that coefficients are Lipschitz continuous on each H_k and that

$$\lim_{k \rightarrow \infty} \tau_{H_k} = \infty.$$

- Since ISDEs like as

$$dX_t^i = dB_t^i + \sum_{j \neq i} \frac{1}{X_t^i - X_t^j} dt \quad (i \in \mathbb{N}).$$

are complicated, it is hard to find out such a sequence of subsets $\{H_k\}$. We give an algorithm to find out such sets by Dirichlet form theory and tail analysis. (In our theorem, exit times do not appear).

Out line of the proof.

Our approach consists of 6 steps:

By the first three steps we construct weak solutions.

By the next three steps we lift them to strong solutions and prove the pathwise uniqueness of ISDEs.

Idea to solve ISDE: From S to $S^{\mathbb{N}}$.

(Step 1) • We start with a random point field μ (a probability measure on configuration space S).

• We construct μ -reversible unlabeled diffusions X by Dirichlet forms.

$$X_t = \sum_{i=1}^{\infty} \delta_{X_t^i}.$$

For this we introduce the map from RPF μ on S to bilinear forms :

$$\mu \mapsto \mathcal{E}^{\mu}(f, g) = \int_S \mathbb{D}[f, g] d\mu \quad \text{on } L^2(S, \mu).$$

Here \mathbb{D} is the standard square field on S :

$$\mathbb{D}[f, g](s) = \frac{1}{2} \sum_{i=1}^{\infty} \frac{\partial \tilde{f}}{\partial s_i} \cdot \frac{\partial \tilde{g}}{\partial s_i}$$

Here f is a local and smooth function on S , and $\tilde{f}(s_1, \dots,)$ is a symmetric function such that $f(s) = \tilde{f}(s_1, \dots,)$, where $s = \sum_{i=1}^{\infty} \delta_{s_i}$.

- If μ is the Poisson RPF $= \Lambda$ with Lebesgue intensity, then the associated diffusion X_t is S -valued Brownian motion $B_t = \sum_{i=1}^{\infty} \delta_{B_t^i}$, which is a reason we call \mathbb{D} the standard square field.

Thus this Dirichlet space is a distorted Brownian motion on S although μ does not have a density with respect to Λ usually.

- We assume:

μ is a Ψ -quasi-Gibbs measure.

Roughly speaking, quasi-Gibbs means that μ has a local density conditioned out side. Gibbs measures are of course quasi-Gibbs, and there exist RPF that are quasi-Gibbs for logarithmic potential Ψ .

- Assume that μ is Ψ -quasi-Gibbs with upper semicontinuous Ψ , and that $\sum_{m=1}^{\infty} m\mu(S_r^m) < \infty$ ($S_r^m = \{s; s(S_r) = m\}$), and that m -density functions on S_r are in $L^2(S_r^m)$ for all $r, m \in \mathbb{N}$. Here $S_r = \{|s| < r\}$.

- With these assumption, the bilinear form is closable and its closure is a quasi-regular Dirichlet form.

- We thus have unlabeled diffusions.

$$S \Rightarrow C([0, \infty); S) \Rightarrow C([0, \infty); S^{\mathbb{N}})$$

(Step 2) • Assuming **non-collision** and **non-explosion** of tagged particles, we can construct labeled dynamics.

• The difficulty to construct $S^{\mathbb{N}}$ -valued diffusion, there is no good measure on $S^{\mathbb{N}}$. (Hence no associated Dirichlet forms).

Even if Brownian motions, the measure should be $dx^{\mathbb{N}}$!

Hence we consider M -Campbell measure $\mu^{[M]}$ of μ .

Introduce the countable family of Dirichlet forms:

$$(\mathcal{E}^{\mu^{[M]}}, L^2(S^M \times S, \mu^{[M]})), \quad \mathbf{X}^{[M]} := (X^{M,1}, \dots, X^{M,M}, \sum_{i=M+1}^{\infty} \delta_{X^{M,i}})$$

There is natural coupling associated diffusions. \Rightarrow

$X^{M,i}$ are independent of M . \Rightarrow

From this consistency we can construct the labeled diffusion on $S^{\mathbb{N}}$.

• We use unlabeled diffusion X_t to couple with these $\mathbf{X}^{[M]}$.

(Step 3) Calculate the logarithmic derivative d^μ . ISDE becomes

$$dX_t^i = dB_t^i + \frac{1}{2}d^\mu(X_t^i, \sum_{j \neq i} \delta_{X_t^j})dt$$

In the case of Ginibre, Sine $_\beta$ (Dyson), Bessel, and Gibbs measures:

$$\beta \nabla \Phi(x) + \beta \lim_{r \rightarrow \infty} \sum_{j \neq i, |x - s_j| < r} \nabla \Psi(x - s_j)$$

Then we have the ISDE (weak solution):

$$dX_t^i = dB_t^i - \frac{\beta}{2} \nabla \Phi(X_t^i) - \frac{\beta}{2} \lim_{r \rightarrow \infty} \sum_{j \neq i, |X_t^i - X_t^j| < r} \nabla \Psi(X_t^i - X_t^j)dt$$

To calculate the logarithmic derivative we use finite particle approximation. In particular, orthogonal polynomials.

The shape of Airy RPF is different.

(Step 4) Introduce:

The infinite system of finite-dimensional SDEs with consistency (IFC):

Let (\mathbf{X}, \mathbf{B}) be a weak solution.

We regard \mathbf{X} as a part of coefficients of SDEs.

For each M consider SDE of $\mathbf{Y}^M = (Y^{M,1}, \dots, Y^{M,M})$:

$$dY_t^{M,i} = dB_t^i - \frac{\beta}{2} \nabla \Phi(Y_t^{M,i}) - \frac{\beta}{2} \sum_{j=1, j \neq i}^M \nabla \Psi(Y_t^{M,i} - Y_t^{M,j}) dt - \frac{\beta}{2} \sum_{j=M+1}^{\infty} \nabla \Psi(Y_t^{M,i} - X_t^j) dt.$$

These (time inhomogeneous, finite-dimensional) SDEs have unique strong solution (under suitable assumptions). Hence

$$\mathbf{Y}^M = \mathbf{X}^M := (X^1, \dots, X^M)$$

- We solve infinite-many finite-dimensional SDEs with consistency in stead of solving a single ISDE.

(Step 5)

- Let $\mathcal{T}_{\text{path}}(S^{\mathbb{N}})$ be the tail σ -field of labeled path space w.r.t. label.

$$\mathcal{T}_{\text{path}}(S^{\mathbb{N}}) = \bigcap_{M=1}^{\infty} \sigma[X^M, \dots,].$$

- \mathbf{Y}^M is a functional of $(\mathbf{B}, (X^{M+1}, \dots,))$.

\Rightarrow If $\lim_{M \rightarrow \infty} \mathbf{Y}^M$ exists, then $\sigma[\mathbf{B}] \vee \mathcal{T}_{\text{path}}(S^{\mathbb{N}})$ -measurable.

\Rightarrow Since $\lim_{M \rightarrow \infty} \mathbf{Y}^M = \mathbf{X}$, \mathbf{X} is $\sigma[\mathbf{B}] \vee \mathcal{T}_{\text{path}}(S^{\mathbb{N}})$ -measurable.

\Rightarrow If $\mathcal{T}_{\text{path}}(S^{\mathbb{N}})$ is trivial, then \mathbf{X} is a strong solution.

- Since we see in the (Step 5) that

$$\mathbf{Y}^M = \mathbf{X}^M := (X^1, \dots, X^M),$$

\mathbf{Y}^M satisfy these.

(Step 6) • We say unlabeled diffusion satisfies the absolutely continuity condition (ACC) if

$$P_\mu(X_t \in \cdot) \prec \mu \quad \text{for all } t.$$

- If ACC is satisfied and if μ is tail trivial, then $\mathcal{T}_{\text{path}}(S^{\mathbb{N}})$ is trivial.
- Tail triviality of RPF \Rightarrow tail triviality of labeled path space.
- $\mathcal{T}_{\text{path}}(S^{\mathbb{N}})$ is the tail σ -field of the labeled path space w.r.t. the label.
- We regard $\mathcal{T}_{\text{path}}(S^{\mathbb{N}})$ as a *boundary condition of ISDE*.
So if its trivial and unique, then the solution of ISDE is unique.
- Our pathwise uniqueness does not exclude the possibility of the existence of a *tail moving* or *shock* solution. It is related to the uniqueness of Dirichlet forms (domain choice).
- We have not yet solve the non-equilibrium problem. We have not yet fully utilize the property of this method, and expect that with this we can solve the non-equilibrium problem at the level of Fritz (1987).

Tail triviality of μ is not a real restriction. Indeed,

Prop 1. *Determinantal RPFs (in continuous spaces) are tail trivial. In particular, Ginibre RPF is tail trivial.*

This result is a generalization of Shirai-Talahashi, and Russel Lyons for discrete spaces.

Note that RPFs appearing in random matrix theory are determinantal random point fields if $\beta = 2$. So our results provide the uniqueness for these.

Even if μ is not tail trivial, we can still apply our results to quasi-Gibbs measures because of the following result.

Prop 2. *Quasi-Gibbs measures μ have decomposition w.r.t. their tail σ -fields $\mathcal{T}(S)$ such that each components are tail trivial: For μ -a.s. s*

$$\mu(A|\mathcal{T}(S))(s) = 1_A(s) \quad \text{for all } A \in \mathcal{T}(S).$$

This is an analogy of the result of Georgii on Gibbs measures on discrete spaces.

END

Weak solutions

2014/9/2/Tue Warwick

UK-Japan Stochastic Analysis School (JSPS Core-to-Core programme)

Workshop dates: 2014-09-01 – 2014-09-05

Outline:

- quasi-Gibbs measures and unlabeled diffusion
- logarithmic derivative and SDE representation
- A sufficient condition for quasi-Gibbs property
- Calculation of logarithmic derivatives
- Examples: Ginibre and Airy RPFs

- We solve ISDEs of the form

$$dX_t^i = dB_t^i + b(X_t^i, \mathbf{X}_t^{\diamond i})dt \quad (i \in \mathbb{N}) \quad (1)$$

Here $\mathbf{X}_t = (X_t^1, \dots,) \in (\mathbb{R}^2)^{\mathbb{N}}$ -valued, and

$$\mathbf{X}_t^{\diamond i} = (X_t^j)_{j \in \mathbb{N} \setminus \{i\}}.$$

The coefficient $b(x, \mathbf{y})$ is symmetric in $\mathbf{y} = (y_i)_{i \in \mathbb{N}}$ for each $x \in \mathbb{R}^2$.

$\mathbf{B}_t = (B_t^1, \dots,)$ is $(\mathbb{R}^2)^{\mathbb{N}}$ -valued standard Brownian motion.

We will construct weak solution (\mathbf{X}, \mathbf{B}) .

Our method can be applied to the case with $\sigma(X_t^i, \mathbf{X}_t^{\diamond i})dB_t^i$.

For simplicity we talk about (1) only.

- Because of the symmetry of $b(x, \mathbf{y})$ in \mathbf{y} , we can rewrite

$$dX_t^i = dB_t^i + b(X_t^i, \mathbf{X}_t^{\diamond i})dt \quad (i \in \mathbb{N}) \quad (2)$$

Here we regard $b(x, \cdot)$ as a function on the configuration space, and

$$\mathbf{X}_t^{\diamond i} = \sum_{j \neq i} \delta_{X_t^j}$$

- We recall the examples: ($i \in \mathbb{N}$)

$$dX_t^i = dB_t^i + \frac{\beta}{2} \lim_{r \rightarrow \infty} \sum_{|X_t^i - X_t^j| < r, j \neq i} \frac{1}{X_t^i - X_t^j} dt \quad (\text{Sine})$$

$$dX_t^i = dB_t^i + \frac{\beta}{2} \lim_{r \rightarrow \infty} \left\{ \left(\sum_{j \neq i, |X_t^j| < r} \frac{1}{X_t^i - X_t^j} \right) - \int_{|x| < r} \frac{\rho(x)}{-x} dx \right\} dt \quad (\text{Airy})$$

$$dX_t^i = dB_t^i + \frac{a}{2X_t^i} dt + \frac{\beta}{2} \sum_{j \neq i}^{\infty} \frac{1}{X_t^i - X_t^j} dt \quad (\text{Bessel})$$

$$dX_t^i = dB_t^i + \lim_{r \rightarrow \infty} \sum_{\substack{|X_t^i - X_t^j| < r \\ j \neq i}} \frac{X_t^i - X_t^j}{|X_t^i - X_t^j|^2} dt \quad (\text{Ginibre})$$

- Gibbsian examples for suitable α and d : ($i \in \mathbb{N}$)

$$dX_t^i = dB_t^i + \frac{\beta}{2} \sum_{j=1, j \neq i}^{\infty} \left\{ \frac{12(X_t^i - X_t^j)}{|X_t^i - X_t^j|^{14}} - \frac{6(X_t^i - X_t^j)}{|X_t^i - X_t^j|^8} \right\} dt \quad (\text{LJ 6-12})$$

$$dX_t^i = dB_t^i + \frac{\beta}{2} \sum_{j \neq i}^{\infty} \frac{X_t^i - X_t^j}{|X_t^i - X_t^j|^\alpha} dt. \quad (\text{Riesz})$$

Cofibration spaces

Set up:

- $S = \mathbb{R}^d$: Space, where particles move,
 - $S_r = \{|x| \leq r\}$,
 - $S = \{s = \sum_i \delta_{s_i}, s(S_r) < \infty (\forall r)\}$:
Configuration space over S .
Polish space with vague topology.
The space of unlabeled particles.
 - $S^{\mathbb{N}}$ is the space of labeled particles.
 - $s = \sum_i \delta_{s_i}$ denotes unlabeled particles.
 $s = (s_i) \in S^{\mathbb{N}}$ denotes labeled particles.
 - Since $S^{\mathbb{N}}$ is too large, we use S instead.
-
- $B_t = \sum_{i=1}^{\infty} \delta_{B_t^i}$ is S -valued Brownian motion.
 - $B_t = (B_t^i)_{i \in \mathbb{N}}$ is $S^{\mathbb{N}}$ -valued Brownian motion.

Canonical square field

For a fun f on S let $f(s) =: \tilde{f}(s_1, \dots)$, where \tilde{f} is symmetric, $s = \sum \delta_{s_i}$.
Let \mathcal{D}_0 be the set of bounded, local, smooth functions f on S .

i.e. f is $\sigma[\pi_r]$ -measurable for some $r < \infty$, \tilde{f} is smooth.

Let \mathbb{D} be the canonical square field on S :

$$\mathbb{D}[f, g](s) = \frac{1}{2} \sum_i \nabla_i \tilde{f} \cdot \nabla_i \tilde{g}.$$

Here $\nabla_i = (\frac{\partial}{\partial s_{i1}}, \dots, \frac{\partial}{\partial s_{id}})$.

The rhs is independent of particular choice of label.

- For a RPF μ we set

$$\mathcal{E}^\mu(f, g) = \int_S \mathbb{D}[f, g] \mu(ds),$$

$$\mathcal{D}_0^\mu = \{f \in \mathcal{D}_0; \mathcal{E}^\mu(f, f) < \infty, f \in L^2(\mu)\}$$

- If we take $\mu = \Lambda$, Poisson RPF with Lebesgue intensity, then the bilinear form associates Brownian motion $B_t = \sum_i \delta_{B_t^i}$.

In this sense \mathbb{D} is the canonical square field.

From RPF to unlabeled diffusion

Outline of the proof:

$$\mu \Rightarrow (\mathcal{E}^\mu, \mathcal{D}_0^\mu, L^2(\mu)) \Rightarrow X_t = \sum_{i=1}^{\infty} \delta_{X_t^i} \Rightarrow \mathbf{X} = (X_t^i)_{i \in \mathbb{N}} \Rightarrow \text{ISDE}$$

- The first arrow is automatic. For a given RPF μ , we can associate a positive bilinear form through the square field \mathbb{D} .
- If $(\mathcal{E}^\mu, \mathcal{D}_0^\mu, L^2(\mu))$ is closable and its closure is quasi-regular, then by Dirichlet form theory an associated μ -reversible diffusion X_t exists.
- For this we introduce a notion of quasi-Gibbs measure.

If μ is quasi-Gibbs with upper semi-continuous potential Ψ , then the bilinear form is closable. In addition, μ satisfies a marginal condition (local boundedness of correlation functions, say), then the form becomes quasi-regular. Hence by the general theory of Dirichlet form there exists the associated unlabeled diffusion X_t .

Quasi-Gibbs measures:

- $\pi_r, \pi_r^c: S \rightarrow S$: projections

$$\pi_r(s) = s(\cdot \cap S_r), \quad \pi_r^c(s) = s(\cdot \cap S_r^c)$$

- For a RPF μ we set

$$\mu_{r,\xi}^m(\cdot) = \mu(\pi_r \in \cdot | s(S_r) = m, \pi_r^c(s) = \pi_r^c(\xi))$$

- Let $\Psi: S \rightarrow \mathbb{R} \cup \{\infty\}$ (interaction).

$$\mathcal{H}_r = \sum_{s_i, s_j \in S_r, i < j} \Psi(s_i - s_j)$$

• • •

Ψ -Quasi-Gibbs meas.

$$\mu_{r,\xi}^m(\cdot) = \mu(\pi_r \in \cdot | s(S_r) = m, \pi_r^c(s) = \pi_r^c(\xi))$$

$$\mathcal{H}_r = \sum_{s_i, s_j \in S_r, i < j} \Psi(s_i - s_j)$$

Def: μ is Ψ -quasi-Gibbs measure if $\exists c_{r,\xi}^m$ s.t.

$$c_{r,\xi}^m^{-1} e^{-\mathcal{H}_r} d\Lambda_r^m \leq \mu_{r,\xi}^m \leq c_{r,\xi}^m e^{-\mathcal{H}_r} d\Lambda_r^m$$

Here $\Lambda_r^m = \Lambda(\cdot | s(S_r) = m)$ and Λ_r is the Poisson RPF with $1_{S_r} dx$.

- The above definition is a simplified version.
- Gibbs measures \Rightarrow Quasi-Gibbs measures: If

$$\mu_{r,\xi}^m = c_r^m e^{-\mathcal{H}_r - \sum_{x_i \in S_r, \xi_j \in S_r^c} \Psi(x_i, \xi_j)} d\Lambda_r^m, \quad (\text{QG})$$

then μ is a canonical Gibbs measure. (QG) does not make sense for

$$\Psi(x, y) = -\log |x - y|$$

Application of quasi-Gibbs property to dynamics

$$\mu \Rightarrow (\mathcal{E}^\mu, \mathcal{D}_0^\mu, L^2(\mu)) \Rightarrow X_t = \sum_{i=1}^{\infty} \delta_{X_t^i} \Rightarrow \mathbf{X} = (X_t^i)_{i \in \mathbb{N}} \Rightarrow \text{ISDE}$$

Unlabeled diffusions

(A1) μ is a Ψ -quasi-Gibbs m with upper-semicont Ψ . \Rightarrow (closability)

(A2) $\sum_{k=1}^{\infty} k\mu(S_r^k) < \infty$, $\sigma_r^k \in L^2(S_r^k, dx) \Rightarrow$ (existence of diffusions)

Here $S_r^k = \{s(S_r) = k\}$, σ_r^k is k -density fun on S_r^k .

Thm 1 (O.'96 (CMP)). (1) (A1) $\Rightarrow (\mathcal{E}^\mu, \mathcal{D}_0^\mu)$ is closable on $L^2(\mu)$.

(2) (A1), (A2) $\Rightarrow \exists$ diffusion $X_t = \sum_i \delta_{X_t^i}$ associated with the closure

$(\mathcal{E}^\mu, \mathcal{D}^\mu)$ of $(\mathcal{E}^\mu, \mathcal{D}_0^\mu)$ on $L^2(\mu)$.

Proof. Outline of (1): Let

$$\mathcal{E}^{\mu_{r,\xi}^m}(f, g) = \int_S \mathbb{D}[f, g] d\mu_{r,\xi}^m.$$

Then $(\mathcal{E}^{\mu_{r,\xi}^m}, \mathcal{D}_0^{\mu_{r,\xi}^m})$ is closable on $L^2(\mu_{r,\xi}^m)$ by (A1).

Hence $(\widehat{\mathcal{E}}_r^\mu, \mathcal{D}_0^\mu)$ are closable on $L^2(\mu)$. Here

$$\widehat{\mathcal{E}}_r^\mu(f, g) = \int_S \frac{1}{2} \sum_{s_i \in S_r} \frac{\partial \check{f}}{\partial s_i} \cdot \frac{\partial \check{g}}{\partial s_i} d\mu \quad (\text{reflecting BC}).$$

By the monotone convergence theorem of closable forms we see

$$\widehat{\mathcal{E}}^\mu(f, f) = \lim_{r \rightarrow \infty} \widehat{\mathcal{E}}_r^\mu(f, f), \quad \widehat{\mathcal{D}}_0 = \{f; \lim_{r \rightarrow \infty} \widehat{\mathcal{E}}_r^\mu(f, f) < \infty\}$$

is closable. Hence $(\mathcal{E}^\mu, \mathcal{D}_0^\mu)$ is closable.

(2) follows from a concrete construction of cut off function. □

Remark 1. In general, the closures of the limit Dirichlet forms

$$(\widehat{\mathcal{E}}^\mu, \widehat{\mathcal{D}}) \quad \text{and} \quad (\mathcal{E}^\mu, \mathcal{D}^\mu)$$

are not equal. We will prove the coincidence of these by using the strong uniqueness of the solutions of the associated ISDEs.

Lang's dynamics ('79) are given by the Dirichlet form $(\widehat{\mathcal{E}}^\mu, \widehat{\mathcal{D}})$.

O's ('96) dynamics are given by $(\mathcal{E}^\mu, \mathcal{D}^\mu)$.

Let $\Psi_2(x, y) = -\log|x - y|$ be the 2-dim Coulomb potential.

Thm 2 (O. AOP '13, O.-Honda (14), O.-Tanemura (14)).

- (1) Ginibre RPF is a $2\Psi_2$ -quasi Gibbs measure.
- (2) Sine $_{\beta}$ RPF are $\beta\Psi_2$ -quasi Gibbs m for $\beta = 1, 2, 4$.
- (3) Bessel $_2^a$ RPF is a $2\Psi_2$ -quasi Gibbs m.
- (4) Airy $_{\beta}$ RPF are $\beta\Psi_2$ -quasi Gibbs m for $\beta = 1, 2, 4$.

$$\mu \Rightarrow (\mathcal{E}^\mu, \mathcal{D}_0^\mu, L^2(\mu)) \Rightarrow X_t = \sum_{i=1}^{\infty} \delta_{X_t^i} \Rightarrow \mathbf{X} = (X_t^i)_{i \in \mathbb{N}} \Rightarrow \text{ISDE}$$

Labeled dynamics

(A1) μ is a Ψ -quasi-Gibbs m with upper-semicont Ψ .

(A2) $\sum_{k=1}^{\infty} k \mu(S_r^k) < \infty$, $\sigma_r^k \in L^2(S_r^k, dx)$

(A3) $\{X_t^i\}$ do not collide each other (non-collision)

(A4) each tagged particle X_t^i never explode (non-explosion)

By (A3) and (A4) the **labeled dynamics**

$$\mathbf{X}_t = (X_t^1, X_t^2, \dots)$$

can be constructed from the unlabeled dynamics

$$X_t = \sum_{i \in \mathbb{N}} \delta_{X_t^i}.$$

Indeed, the particles keep the initial label forever.

Sufficient condition of (A3) & (A4)

Let $S_{s,i} = S_s \cap S_i$:

$$S_s = \{s \in S; s(\{x\}) = 0 \text{ for all } x \in S\}, \quad S_i = \{s \in S; s(S) = \infty\}.$$

- (A3) is equivalent to

$$\text{Cap}^\mu(S_{s,i}^c) = 0. \quad (3)$$

Let ρ^n be a n -correlation function of μ .

Lem 1. *Suppose μ is quasi-Gibbs with Ψ . Let ρ^2 be 2-correlation function of μ . Suppose one of the following holds. Then (A3) holds.*

(1) $d \geq 2$ and ρ^2 are locally bounded.

(2) $d = 1$ and

$$\rho^2(x, y) \leq Ch(|x - y|) \text{ locally near } \{x = y\}.$$

Here $h(t)$ such that

$$\int_{0+}^1 \frac{1}{h(t)} dt = \infty.$$

Corollary 1. *Sine $_\beta$, Airy $_\beta$, Bessel $_\beta$ ($\beta \geq 1$), Ginibre RPFs satisfy (A2).*

General theorems on infinite-dim SDEs

- By (A3) we represent one-labeled process $(X_t^1, \sum_{j=2}^{\infty} \delta_{X_t^j})$ by the Dirichlet space

$$(\mathcal{E}^{\mu^{[1]}}, \mathcal{D}^{\mu^{[1]}}, L^2(\mu^{[1]})).$$

Applying Takeda criteria based on Lyons-Zheng decomposition we deduce (A4) from $\exists T > 0$

$$\liminf_{r \rightarrow \infty} \left\{ \int_{|x| \leq r+R} \rho^1(x) dx \right\} \left\{ \int_{\frac{r}{\sqrt{(r+R)T}}} g(u) du \right\} = 0 \quad \text{for all } T. \quad (4)$$

Lem 2. (A4) follows from (4).

SDE representation

$$\mu \Rightarrow (\mathcal{E}^\mu, \mathcal{D}_0^\mu, L^2(\mu)) \Rightarrow X_t = \sum_{i=1}^{\infty} \delta_{X_t^i} \Rightarrow \mathbf{X} = (X_t^i)_{i \in \mathbb{N}} \Rightarrow \text{ISDE}$$

ISDE representation

Log derivative of μ : precise correspondence between RPFs & potentials

- Let μ_x be the (reduced) Palm m. of μ conditioned at x

$$\mu_x(\cdot) = \mu(\cdot - \delta_x | s(x) \geq 1)$$

- Let μ^1 be the 1-Campbell measure on $\mathbb{R}^d \times S$:

$$\mu^1(A \times B) = \int_A \rho^1(x) \mu_x(B) dx$$

- $d\mu \in L^1_{loc}(\mathbb{R}^d \times S, \mu^1)$ is called the **log derivative** of μ if

$$\int_{\mathbb{R}^d \times S} \nabla_x f d\mu^1 = - \int_{\mathbb{R}^d \times S} f d\mu d\mu^1 \quad \forall f \in C_0^\infty(\mathbb{R}^d) \otimes \mathcal{D}_0$$

Here ∇_x is the nabla on \mathbb{R}^d .

- Very informally

$$d\mu = \nabla_x \log \mu^1$$

- A calculation of log derivative of Gibbs measures are trivial. Indeed, it is immediate from DLR equation.
- This is not the case for RPFs appearing in RMT. We will give a sufficient condition later.

Log derivative

A very informal calculation shows:

- If $\mu^1(dx ds) = m(x, s_1, \dots) dx \prod_i ds_i$, then

$$\begin{aligned} & - \int \nabla_x f(x, s_1, \dots) \mu^1(dx ds_1 \cdots) \\ &= - \int \nabla_x f(x, s_1, \dots) m(x, s_1, \dots) dx \prod_i ds_i \\ &= \int f(x, s_1, \dots) \nabla_x m(x, s_1, \dots) dx \prod_i ds_i \\ &= \int f(x, s_1, \dots) \frac{\nabla_x m(x, s_1, \dots)}{m(x, s_1, \dots)} m(x, s_1, \dots) dx \prod_i ds_i. \end{aligned}$$

Hence

$$d\mu = \frac{\nabla_x m(x, s_1, \dots)}{m(x, s_1, \dots)} = \nabla_x \log m(x, s_1, \dots).$$

General theorems on infinite-dim SDEs

(A1) μ is a Ψ -quasi-Gibbs m with upper-semicont Ψ .

(A2) $\sum_{k=1}^{\infty} k\mu(S_r^k) < \infty$, $\sigma_r^k \in L^2(S_r^k, dx)$

(A3) $\{X_t^i\}$ do not collide each other

(A4) each tagged particle X_t^i never explode

(A5) The log derivative $d^\mu \in L_{loc}^1(\mu^1)$ exists \Rightarrow (SDE representation)

General theorems on infinite-dim SDEs

(A1) μ is a Ψ -quasi-Gibbs m with upper-semicont Ψ .

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Thm 3. (O.12(PTRF)) (A1)–(A5) $\Rightarrow \exists S_0 \subset S$ such that $\mu(S_0) = 1$, and that, for $\forall s \in u^{-1}(S_0)$, there exists a solution (\mathbf{X}, \mathbf{B}) satisfying

$$dX_t^i = dB_t^i + \frac{1}{2} d^\mu(X_t^i, \sum_{j \neq i} \delta_{X_t^j}) dt, \quad (X_0^i)_{i \in \mathbb{N}} = s$$

$$\mathbf{X}_t \in u^{-1}(S_0) \quad \text{for all } t$$

Here $u: S^{\mathbb{N}} \rightarrow S$ such that $u((s_i)) = \sum_i \delta_{s_i}$.

Corollary 2. Suppose that there exists a RPF μ satisfying (A1)–(A4) and

$$\nabla_x \log \mu^{[1]}(x, s) = 2b(x, s).$$

Then ISDE (1) has a weak solution.

General theorems on infinite-dim SDEs

Proof:

• $S^{\mathbb{N}}$ does not have good measures \Rightarrow no Dirichlet forms on $S^{\mathbb{N}}$ \Rightarrow Introduce a sequence of spaces with Campbell measures $\mu^{[M]}$:

$$S^M \times S, \quad d\mu^{[M]} = \rho^M(\mathbf{x}_M) \mu_{\mathbf{x}_M}(ds) d\mathbf{x}_M$$

Here ρ^M is a M -correlation function of μ and $\mu_{\mathbf{x}_M}$ is the reduced Palm measure conditioned at \mathbf{x}_M .

Let $\mathbb{D}^{[M]}$ be the natural square field of $S^M \times S$. Let

$$\mathcal{E}^{[M]}(f, g) = \int_{S^M \times S} \mathbb{D}^{[M]}[f, g] d\mu^{[M]},$$

$$L^2(\mu^{[M]}), \quad C_0^\infty(S^M) \otimes \mathcal{D}_\circ.$$

Lem 3. *These bilinear forms are closable, and their closures are quasi-regular Dirichlet forms. Hence associated diffusion $(\mathbf{X}_t^M, \mathbf{X}_t^{M*})$ exists:*

$$(\mathbf{X}_t^M, \mathbf{X}_t^{M*}) = (X_t^{M,1}, \dots, X_t^{M,M}, \sum_{i=M+1}^{\infty} \delta_{X_t^{M,i}})$$

Coupling of Dirichlet forms:

- Let fix a label ℓ . Let

$$X_t = \sum_{i=1}^{\infty} \delta_{X_t^i}$$

be the unlabeled diffusion associated with the original unlabeled Dirichlet form

$$(\mathcal{E}^\mu, \mathcal{D}^\mu, L^2(\mu)).$$

Thm 4. *Associated diffusions have consistency*

$(X_t^{M,1}, \dots, X_t^{M,M}, X_t^{M,M+1}, \dots) = (X_t^1, \dots, X_t^M, X_t^{M+1}, \dots)$ *in law or equivalently*

$$(\mathbf{X}_t^M, \mathbf{X}_t^{M*}) = (X_t^1, \dots, X_t^M, \sum_{i=M+1}^{\infty} \delta_{X_t^i}) \quad \text{in law}$$

From this coupling and Fukushima decomposition (Itô formula) we prove that (X_t^i) satisfies the ISDE. We use the M -labeled process $(\mathbf{X}_t^M, \mathbf{X}_t^{M*})$, to apply Itô formula to coordinate functions x_1, \dots, x_M .

Coupling of Dirichlet forms:

- The key point here is that, instead of large space

$$S^{\mathbb{N}}$$

we use a system of countably infinite *good* infinite dimensional space

$$S^1 \times S, S^2 \times S, S^3 \times S, S^4 \times S, S^5 \times S, S^6 \times S, S^7 \times S, \dots$$

- By the diffusion X on the original unlabeled space

$$S,$$

we construct a coupling of diffusions (\mathbf{X}^M, X^{M*}) on these infinite many spaces $S^M \times S$.

- From this coupling, we have the ISDE representation. Indeed, we can apply Itoô formula to each coordinate functions $f(\mathbf{x}) = x_k$. We use $\mathcal{E}^{[M]}(f, g)$ for $1 \leq k \leq M$.

Log derivative of μ : precise correspondence between RPFs & potentials

- The log derivative gives the precise correspondence between RPFs μ and potentials (Φ, Ψ) .
- We next give examples of logarithmic derivatives

$$d^\mu = \nabla_x \log \mu^1$$

Thm 5 (O. PTRF 12).

(1) Let μ_{gin} be the Ginibre RPF. Then

$$d^{\mu_{\text{gin}}}(x, s) = \lim_{r \rightarrow \infty} 2 \sum_{|x-s_i| < r} \frac{x - s_i}{|x - s_i|^2}$$

$$d^{\mu_{\text{gin}}}(x, s) = -2x + \lim_{r \rightarrow \infty} 2 \sum_{|s_i| < r} \frac{x - s_i}{|x - s_i|^2}$$

(2) Let $\mu_{\text{sin}, \beta}$ be the Sine $_\beta$ RPF. Suppose $\beta = 1, 2, 4$. Then

$$d^{\mu_{\text{sin}, \beta}}(x, s) = \lim_{r \rightarrow \infty} \beta \sum_{|x-s_i| < r} \frac{1}{x - s_i}$$

Thm 6 (O.-Honda). Let $\mu_{\text{bes}, 2}^a$ be the Bessel $_2^a$ RPF. Then

$$d^{\mu_{\text{bes}, 2}^a}(x, s) = \frac{a}{x} + 2 \sum_{|x-s_i| < r} \frac{1}{x - s_i}$$

Thm 7 (O.-Tanemura). [*Airy RPFs: $\mu_{\text{Ai},\beta}$*]

Let $\beta = 1, 2, 4$. Then the log derivative $d^{\mu_{\text{Ai},\beta}}$ is

$$d^{\mu_{\text{Ai},\beta}}(x, s) = \beta \lim_{r \rightarrow \infty} \left\{ \left(\sum_{|x-s_i| < r} \frac{1}{x - s_i} \right) - \int_{|x| < r} \frac{\varrho(x)}{-x} dx \right\}$$

Here

$$\varrho(x) = \frac{\sqrt{-x}}{\pi} \mathbf{1}_{(-\infty, 0)}(x)$$

A criteria of Quasi-Gibbs property.

For $\Phi : S \rightarrow \mathbb{R} \cup \{\infty\}$ and $\Psi : S \times S \rightarrow \mathbb{R} \cup \{\infty\}$, let

$$\mathcal{H}_A^{\Phi, \Psi}(x) = \sum_{x_i \in A} \Phi(x_i) + \sum_{x_i, x_j \in A, i < j} \Psi(x_i, x_j), \quad \text{where } x = \sum_i \delta_{x_i}.$$

We assume $\Phi < \infty$ almost everywhere (a.e.) to avoid triviality.

We set

$$\mathcal{H}_r(x) = \mathcal{H}_{S_r}^{\Phi, \Psi}(x). \quad (5)$$

For a subset $A \subset S$, we define the map $\pi_A : S \rightarrow S$ by $\pi_A(s) = s(A \cap \cdot)$.

Let Λ be the Poisson RPF for which the intensity is the Lebesgue measure on S . We set

$$\Lambda_r = \Lambda(\cdot \cap S_r^m).$$

We write $\nu_1 \leq \nu_2$ if $\nu_1(A) \leq \nu_2(A)$ for all $A \in \mathcal{B}$. Here ν_1, ν_2 are measures on (Ω, \mathcal{B}) .

A criteria of Quasi-Gibbs property

Definition 1. A RPF μ is called a (Φ, Ψ) -quasi-Gibbs measure if
(1) There exists an increasing sequence $\{b_r\} \subset \mathbb{N}$ such that, for each $r, m \in \mathbb{N}$, there exists a sequence of Borel subsets $S_{r,k}^m$ satisfying

$$S_{r,k}^m \subset S_{r,k+1}^m \subset S_r^m \text{ for all } k, \quad (6)$$

$$\lim_{k \rightarrow \infty} \mu_{r,k}^m = \mu_r^m \text{ weakly,} \quad (7)$$

where $\mu_{r,k}^m = \mu(\cdot \cap S_{r,k}^m)$ and $\mu_r^m = \mu(\cdot \cap S_r^m)$.

(2) For all $r, m, k \in \mathbb{N}$ and $\mu_{r,k}^m$ -a.e. $s \in S$,

$$\frac{1}{C} e^{-\mathcal{H}_r(x)} \mathbf{1}_{S_r^m}(x) \Lambda_r^m(dx) \leq \mu_{r,k,s}^m(dx) \leq C e^{-\mathcal{H}_r(x)} \mathbf{1}_{S_r^m}(x) \Lambda_r^m(dx). \quad (8)$$

Here, $C = C(r, m, k, \pi_{S_r^c}(s))$ is a positive constant and $\mu_{r,k,s}^m$ is the regular conditional probability measure of $\mu_{r,k}^m$ defined as

$$\mu_{r,k,s}^m(dx) = \mu_{r,k}^m(\pi_{S_r} \in dx \mid \pi_{S_r^c}(s)). \quad (9)$$

A criteria of Quasi-Gibbs property

We give a set of conditions for the quasi-Gibbs property.

(H.1) The measure μ has a locally bounded, n -correlation function ρ^n for each $n \in \mathbb{N}$.

(H.2) \exists probability measures $\{\mu^N\}_{N \in \mathbb{N}}$ on S such that:

(1) The n -correlation functions ρ_N^n of μ^N satisfy

$$\lim_{N \rightarrow \infty} \rho_N^n(\mathbf{x}_n) = \rho^n(\mathbf{x}_n) \quad \text{a.e.} \quad \text{for all } n \in \mathbb{N}, \quad (10)$$

$$\sup\{\rho_N^n(\mathbf{x}_n); N \in \mathbb{N}, \mathbf{x}_n \in S_r^n\} \leq \{Cn^\delta\}^n \quad \text{for all } n, r \in \mathbb{N}, \quad (11)$$

where $C = C(r) > 0$, and $\delta = \delta(r) < 1$.

(2) $\mu^N(s(S) = n_N) = 1$ for each N , where $n_N \uparrow \in \mathbb{N}$.

[A good finite-particle approximation $\{\mu^N\}_{N \in \mathbb{N}}$]

(3) μ^N is a (Φ^N, Ψ^N) -canonical Gibbs measure.

(4) There exists a sequence $\{m_\infty^N\}_{N \in \mathbb{N}}$ in \mathbb{R}^d such that

$$\lim_{N \rightarrow \infty} \{\Phi^N(x) - m_\infty^N \cdot x\} = \Phi(x) \quad \text{for a.e. } x, \quad (12)$$

$$\inf_{N \in \mathbb{N}} \inf_{x \in S} \{\Phi^N(x) - m_\infty^N \cdot x\} > -\infty.$$

(5) The interaction potentials $\Psi^N : S \times S \rightarrow \mathbb{R} \cup \{\infty\}$ satisfies

$$\lim_{N \rightarrow \infty} \Psi^N = \Psi \text{ compactly and uniformly in } C^1(S \times S \setminus \{x = y\}), \quad (13)$$

$$\inf_{N \in \mathbb{N}} \inf_{x, y \in S_r} \Psi^N(x, y) > -\infty \quad \text{for all } r \in \mathbb{N}.$$

[Airy RPF (soft edge scaling limit)]

Remark 2.

- For the GUE soft-edge (the Airy RPF), we take $m_\infty^N = N^{1/3}$.
- In fact, in this case, the limit of Φ^N diverges.
- Hence, we substitute $m_\infty^N \cdot x$ from $\Phi^N(x)$ to make the limit finite.
- We see that the terms $m_\infty^N \cdot x$ are cancelled by the interaction terms.

$$-\frac{\beta}{4} \sum_{i=1}^N |2\sqrt{N} + N^{-1/6}x_i|^2 = -\frac{\beta}{4} \sum_{i=1}^N \{4N + N^{-1/3}|x_i|^2 + 4N^{1/3}x_i\}.$$

[Ψ -tightness]

- The next assumption (H.3) is a tightness condition on $\{\mu^N\}$ according to the interaction Ψ^N .
- Let $x = \sum \delta_{x_i}$, $y = \sum \delta_{y_j} \in S$, $S_{rs} = S_s \setminus S_r$, and $S_{r\infty} = S_r^c$. For $r < s \leq t < u \leq \infty$, we set

$$\Psi_{rs,tu}^N(x, y) = \sum_{x_i \in S_{rs}, y_j \in S_{tu}} \Psi^N(x_i, y_j) \quad . \quad (14)$$

We write $\Psi_{0r,rs}^N(x, y) = \Psi_{0r,rs}^N(x, y)$ if $x = \delta_x$.

$$\tilde{\Psi}_{rs,tu}^N(x, y) = \Psi_{rs,tu}^N(x, y) + \left\{ \sum_{x_i \in S_{rs}} x_i \right\} \cdot (m_t^N - m_u^N). \quad (15)$$

For $\{\Psi^N\}$, $r, k \in \mathbb{N}$, and $\{m_s^N\}$

$$H_{r,k}^N = \left\{ y \in S; y(S) = n_N, \left\{ \sup_{r < s \in \mathbb{N}} \sup_{\substack{x, w \in S_r \\ x \neq w}} \frac{|\tilde{\Psi}_{0r,rs}^N(x, y) - \tilde{\Psi}_{0r,rs}^N(w, y)|}{|x - w|} \right\} \leq k \right\}.$$

[A sufficient condition of quasi-Gibbs property]

We define $H_{r,k}$ as

$$H_{r,k} = \sum_{N=1}^{\infty} H_{r,k}^N. \quad (16)$$

(H.3) There exists a sequence $\{m_s^N\}$ in \mathbb{R}^d such that the set $H_{r,k}$ satisfies the following:

$$\lim_{k \rightarrow \infty} \limsup_{N \rightarrow \infty} \mu^N(H_{r,k}^c) = 0 \quad \text{for all } r \in \mathbb{N}, \quad (17)$$

$$\lim_{s \rightarrow \infty} m_s^N = m_{\infty}^N, \quad (18)$$

$$\sup_{N \in \mathbb{N}} |m_s^N| < \infty \quad \text{for all } s \in \mathbb{N}. \quad (19)$$

Thm 8. Assume (H.1), (H.2) and (H.3). Then μ is a (Φ, Ψ) -quasi-Gibbs measure.

$[d = 1, 2]$

- We next assume $d = 1, 2$. To unify these two cases, we set $S = \mathbb{C}$ and regard here \mathbb{R}^2 as \mathbb{C} .

- We assume Ψ^N is independent of N and of the form

$$\Psi(x, y) := \Psi^N(x, y) = -\beta \log |x - y| \quad (\beta \in \mathbb{R}). \quad (20)$$

- We give a suff condition of (H.3) through correlation functions.

- Let $x = \sum_i \delta_{x_i}$ and $\tilde{S}_{rs} = \tilde{S}_s \setminus S_r$, where $S_r = \{s \in S; |s| < r\}$, as before. For $1 \leq r < s \leq \infty$ let $v_{\ell, rs} : S \rightarrow \mathbb{C}$ such that

$$v_{\ell, rs}(x) = \beta \left\{ \sum_{x_i \in \tilde{S}_{rs}} \frac{1}{x_i^\ell} \right\} \quad (\ell \geq 2) \quad (21)$$

$$v_{1, rs}^N(x) = \beta \left\{ \sum_{x_i \in \tilde{S}_{rs}} \frac{1}{x_i} \right\} + \bar{m}_r^N - \bar{m}_s^N \quad (\ell = 1). \quad (22)$$

Here $\bar{m}_r^N = m_{r,1}^N - \sqrt{-1}m_{r,2}^N$ is the complex conjugate of m_r^N .

[d=1,2]

Now the key assumption is as follows.

(H.4) There exists an ℓ_0 such that $2 \leq \ell_0 \in \mathbb{N}$ and that

$$\sup_{N \in \mathbb{N}} \left\{ \int_{1 \leq |x| < \infty} \frac{1}{|x|^{\ell_0}} \rho_N^1(x) dx \right\} < \infty \quad (23)$$

and that, for each $1 < \ell < \ell_0$,

$$\sup_{N \in \mathbb{N}} \|v_{\ell,rs}\|_{L^1(\mu^N)} < \infty \text{ for all } r < s \in \mathbb{N}, \quad (24)$$

$$\lim_{s \rightarrow \infty} \sup_{N \in \mathbb{N}} \|v_{\ell,s\infty}\|_{L^1(\mu^N)} = 0 \quad (25)$$

and that, for each $\ell = 1$,

$$\sup_{N \in \mathbb{N}} \left\| \sup_{M \in \mathbb{N}} v_{\ell,rs}^M \right\|_{L^1(\mu^N)} < \infty \text{ for all } r < s \in \mathbb{N}, \quad (26)$$

$$\lim_{s \rightarrow \infty} \sup_{N \in \mathbb{N}} \left\| \sup_{M \in \mathbb{N}} v_{\ell,s\infty}^M \right\|_{L^1(\mu^N)} = 0. \quad (27)$$

Thm 9. Assume (20) and $S = \mathbb{C}$. Assume (H.1), (H.2) and (H.4). Assume (18). Then μ is a (Φ, Ψ) -quasi-Gibbs measure.

Calculation of logarithmic derivative

- Assume that n -point cor funs $\{\rho^{N,n}\}$ satisfy for each $r, n \in \mathbb{N}$

$$\lim_{N \rightarrow \infty} \rho^{N,n}(\mathbf{x}) = \rho^n(\mathbf{x}) \quad \text{uniformly on } S_r^n, \quad (28)$$

$$\sup_{N \in \mathbb{N}} \sup_{\mathbf{x} \in S_r^n} \rho^{N,n}(\mathbf{x}) \leq C_1^{-n} n^{C_2 n}, \quad 0 < C < \infty, 0 < C_2 < 1, . \quad (29)$$

Calculation of logarithmic derivative

- We assume that μ^N have log derivative d^N such that

$$d^N(x, y) = u^N(x) + g_s^N(x, y) + w_s^N(x, y) \quad (30)$$

Here $g, g^N, v, v^N : S^2 \rightarrow \mathbb{R}^d$ and $w : S \rightarrow \mathbb{R}^d$ and set $(y = \sum_i \delta_{y_i})$

$$g_s(x, y) = \int_{|x-y| < s} v(x, y) dy + \sum_{|x-y_i| < s} g(x, y_i),$$

$$g_s^N(x, y) = \int_{|x-y| < s} v^N(x, y) dy + \sum_{|x-y_i| < s} g^N(x, y_i),$$

$$w_s^N(x, y) = \int_{s \leq |x-y|} v^N(x, y) dy + \sum_{s \leq |x-y_i|} g^N(x, y_i) \in L_{\text{loc}}^{\hat{p}}(\mu^1).$$

Calculation of logarithmic derivative

- Let $1 < p < \hat{p} < \infty$. Assume that

$$\limsup_{N \rightarrow \infty} \int_{S_r \times \mathcal{S}} |d^N - u^N|^{\hat{p}} d\mu^{N,1} < \infty \quad \text{for all } r \in \mathbb{N} \quad (31)$$

$$\lim_{N \rightarrow \infty} u^N = u \quad \text{in } L_{\text{loc}}^{\hat{p}}(S, dx) \quad (32)$$

$$\lim_{N \rightarrow \infty} g_s^N = g_s \quad \text{in } L_{\text{loc}}^{\hat{p}}(\mu^1) \quad \text{for all } s, \quad (33)$$

$$\lim_{s \rightarrow \infty} \limsup_{N \rightarrow \infty} \int_{S_r \times \mathcal{S}} |w_s^N(x, y) - w(x)|^{\hat{p}} d\mu^{N,1} = 0. \quad (34)$$

Recall that

$$g_s(x, y) = \int_{|x-y| < s} v(x, y) dy + \sum_{|x-y_i| < s} g(x, y_i)$$

Thm 10. Assume (28)–(34). Then d^μ exists in $L_{\text{loc}}^p(\mu^1)$ given by

$$d^\mu(x, y) = u(x) + \lim_{s \rightarrow \infty} g_s(x, y) + w(x). \quad (35)$$

Calculation of logarithmic derivative

Recall that

$$g_s(x, y) = \int_{|x-y|<s} v(x, y) dy + \sum_{|x-y_i|<s} g(x, y_i)$$

Thm 10 The log derivative d^μ exists in $L^p_{\text{loc}}(\mu^1)$ and is given by

$$d^\mu(x, y) = u(x) + \lim_{s \rightarrow \infty} g_s(x, y) + w(x). \quad (36)$$

Example 1. In the case of Ginibre RPF, we take

$$\begin{aligned} u^N(x) &= u(x) = -2x, & w(x) &= 2x, \\ v^N(x, y) &= v(x, y) = 0, \\ g^N(x, y) &= g(x, y) = \frac{2(x-y)}{|x-y|^2}. \end{aligned}$$

Calculation of logarithmic derivative

Example 2. In the case of Airy RPF, we take

$$u^N(x) = \beta \left\{ \int_{\mathbb{R}} \frac{\rho_{\beta,x}^{N,1}(y)}{x-y} dy \right\} - N^{1/3} - \frac{N^{-1/3}}{2} x$$

$$u(x) = \beta \lim_{s \rightarrow \infty} \left\{ \int_{|s| < s} \frac{\rho_{\beta,x}^1(y)}{x-y} dy - \int_{|y| < s} \frac{\varrho(y)}{-y} dy \right\}$$

$$w(x) = 0$$

$$v^N(x, y) = -\beta \frac{\rho_{\beta,x}^{N,1}(y)}{x-y}$$

$$v(x, y) = -\beta \frac{\rho_{\beta,x}^1(y)}{x-y}$$

$$g^N(x, y) = g(x, y) = \frac{\beta}{x-y}.$$

Strong solutions and pathwise uniqueness

2014/9/1/Mon–2014/9/5/Wed Warwick

UK-Japan Stochastic Analysis School (JSPS Core-to-Core programme)

Outline:

- Unique strong solutions of ISDEs (general theorems)
- Triviality of tail σ -fields of labeled path spaces.
- Applications to interacting Brownian motions in infinite dimensions.

General theorems on infinite-dim SDEs

- (A1) μ is a Ψ -quasi-Gibbs m with upper-semicont Ψ .
- (A2) $\sum_{k=1}^{\infty} k\mu(S_r^k) < \infty$, $\sigma_r^k \in L^2(S_r^k, dx)$
- (A3) $\{X_t^i\}$ do not collide each other
- (A4) each tagged particle X_t^i never explode
- (A5) The log derivative $d^\mu \in L_{loc}^1(\mu^1)$ exists

General theorems on infinite-dim SDEs

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(A5) The log derivative $d^\mu \in L_{loc}^1(\mu^1)$ exists

Thm 1. (O.12(PTRF)) (A1)–(A5) $\Rightarrow \exists S_0 \subset S$ such that

$$\mu(S_0) = 1,$$

and that, for $\forall s \in u^{-1}(S_0)$, $\exists u^{-1}(S_0)$ -valued pr. $(X_t^i)_{i \in \mathbb{N}}$ and $\exists S^{\mathbb{N}}$ -valued Brownian m. $(B_t^i)_{i \in \mathbb{N}}$ satisfying

$$dX_t^i = dB_t^i + \frac{1}{2} d^\mu(X_t^i, \sum_{j \neq i} \delta_{X_t^j}) dt, \quad (X_0^i)_{i \in \mathbb{N}} = s \quad (1)$$

Here $u: S^{\mathbb{N}} \rightarrow S$ such that $u((s_i)) = \sum_i \delta_{s_i}$.

- The solution (\mathbf{X}, \mathbf{B}) is **not** a strong solution.
- In this talk we construct a strong solution from a weak solution, and prove pathwise uniqueness.

Tail σ -field of configuration space s

- To construct strong solutions, we use two geometric properties of RPFs. : Tail triviality & Tail decomposition
- Let $\pi_r^c: S \rightarrow S$ such that $\pi_r^c(s) = s(\cdot \cap S_r^c)$, where $S_r = \{|s| < r\}$.
- Let $\mathcal{T} = \mathcal{T}(S)$ be the tail σ field of S :

$$\mathcal{T}(S) = \bigcap_{r=1}^{\infty} \sigma[\pi_r^c].$$

1:tail

Thm 2, *Let μ be a determinantal RPF. Then $\mathcal{T}(S)$ is μ -trivial.*

- Thm 2 is a generalization of the result for the discrete determinantal RPFs due to Russel Lyons, Shirai-Takahashi.
- In general, quasi-Gibbs measures μ are not tail trivial. Hence we introduce the tail decomposition of μ .



Tail triviality of determinantal RPFs & Tail decomp of quasi-Gibbs m

Let $\mathcal{T} = \mathcal{T}(S)$ be the tail σ field of S as above.

Let $\mu(\cdot|\mathcal{T})$ be the regular conditional probability.

Then by construction

$$\mu(\cdot) = \int_S \mu(\cdot|\mathcal{T})(\xi) \mu(d\xi)$$

and, for any $A \in \mathcal{T}$,

$$\mu(A|\mathcal{T})(\xi) = 1_A(\xi) \quad \text{for } \mu\text{-a.s. } \xi.$$

We can interchange the roll of “for any $A \in \mathcal{T}$ ” and “for μ -a.s. ξ ”.

1:decom

Thm 3. Let μ be a quasi-Gibbs measure. Then for μ -a.s. ξ ,

$$\mu(A|\mathcal{T})(\xi) = 1_A(\xi) \quad \text{for any } A \in \mathcal{T}.$$

1:decom

- Thm 3 is a generalization of the result for the discrete Gibbs m due to Georgii.
- With this, the assumption of tail triviality of μ turn out to be not an essential restriction.

existence of strong solution

$$dX_t^i = dB_t^i + \frac{1}{2}d^\mu(X_t^i, \sum_{j \neq i} \delta_{X_t^j})dt, \quad (X_0^i)_{i \in \mathbb{N}} = s$$

We introduce the condition such that the drifts $d^\mu(x, s)$ are locally Lipschitz continuous in x for fixed outside $\pi_r^c(s)$.

Let $S_r = \{|x| < r\}$ and

$$H(r, n) = \{s = \sum_i \delta_{s_i}; |\nabla_x d^\mu(s_i, s - \delta_{s_i})| < n \text{ for } \forall i \text{ s.t. } s_i \in S_r\},$$

$$H = \bigcap_{r=1}^{\infty} \bigcup_{n=1}^{\infty} H(r, n).$$

(A6) $\text{Cap}^\mu(H^c) = 0$ + marginal assumption

- We pose in (A6) a condition that the coefficients $d^\mu(x, X_t^{i\blacklozenge})$ in x are Lipschitz continuous in each $H(r, n)$. Here $X_t^{i\blacklozenge} = \sum_{j \neq i} \delta_{X_t^j}$.

existence of strong solution

(A1) μ is a Ψ -quasi-Gibbs m with upper-semicont Ψ .

(A2) $\sum_{k=1}^{\infty} k\mu(S_r^k) < \infty$, $\sigma_r^k \in L^2(S_r^k, dx)$

(A3) $\{X_t^i\}$ do not collide each other

(A4) each tagged particle X_t^i never explode

(A5) The log derivative $d^\mu \in L_{loc}^1(\mu^1)$ exists

1:strong (A6) $\text{Cap}^\mu(H^c) = 0$.

Thm 4 (O.-Tanemura). (A1)–(A6). \Rightarrow (1) *The ISDE*

$$dX_t^i = dB_t^i + \frac{1}{2}d^\mu(X_t^i, \sum_{j \neq i} \delta_{X_t^j})dt, \quad (X_0^i)_{i \in \mathbb{N}} = s$$

has a strong solution for $s = (s_i) \in S^{\mathbb{N}}$ s.t. $\sum_i \delta_{s_i} \in H$.



existence of strong solution

(A1) μ is a Ψ -quasi-Gibbs m with upper-semicont Ψ .

(A2) $\sum_{k=1}^{\infty} k\mu(S_r^k) < \infty$, $\sigma_r^k \in L^2(S_r^k, dx)$

(A3) $\{X_t^i\}$ do not collide each other

(A4) each tagged particle X_t^i never explode

(A5) The log derivative $d\mu \in L_{loc}^1(\mu^1)$ exists

(A6) $\text{Cap}_{1:\text{strong}}^{\mu}(H^c) = 0$.

Thm 4[O.-Tanemura] (A1)–(A6). \Rightarrow (1) The ISDE

$$dX_t^i = dB_t^i + \frac{1}{2}d\mu(X_t^i, \sum_{j \neq i} \delta_{X_t^j})dt, \quad (X_0^i)_{i \in \mathbb{N}} = s$$

has a strong solution for $s = (s_i) \in S^{\mathbb{N}}$ s.t. $\sum_i \delta_{s_i} \in H$.

(2) The ass unlabeled diffusion $X = \sum_i \delta_{X^i}$ satisfies

$$P_{\mu_{\xi}} \circ X_t^{-1} \prec \mu_{\xi} \quad (\forall t) \quad \text{for } \mu\text{-a.s. } \xi$$

1:decom

Here $\mu_{\xi} = \mu(\cdot | \mathcal{T}(S))(\xi)$ in Thm 3.

Decomposition of unlabeled state space of strong solutions 1

- By construction $\mu(\cdot|\xi)(A)$ are \mathcal{T} -measurable in ξ for each $A \in \mathcal{B}(S)$. 1:decom
- By Thm 3, we take a version of $\mu(\cdot|\xi)$ such that, for μ -a.s. $a \in S$. :22w

$$\mu(\cdot|\xi)(A) = 1_A(a) \quad \text{for all } A \in \mathcal{T}. \quad (2)$$

- Let $\sim_{\mathcal{T}}$ be the equivalence relation such that $a \sim_{\mathcal{T}} b$ if and only if :22x

$$1_A(a) = 1_A(b) \quad \text{for all } A \in \mathcal{T}. \quad (3)$$

- From (2) we deduce that the set H in Thm 4 can be decomposed as a disjoint sum :22w

$$H = \sum_{[\xi] \in H/\sim_{\mathcal{T}}} H^{\xi} \quad \text{such that} \quad \mu(\cdot|\xi)(H^{\xi}) = 1. \quad (4) \quad \text{:22y}$$

The solution in Thm 4 satisfy for μ_{ξ} -a.s. $s \in H^{\xi}$ 1:strong

$$P_s(X_t \in H^{\xi} \text{ for all } t) = 1.$$

Uniqueness of strong solutions 1

1:strong0

Thm 5 (O.-Tanemura). Assume (A1)–(A6).

Let $\mathbf{X} = (X^i)$ and $\widehat{\mathbf{X}} = (\widehat{X}^i)$ be strong sol of the ISDE

$$dX_t^i = dB_t^i + \frac{1}{2}d^\mu(X_t^i, \sum_{j \neq i} \delta_{X_t^j})dt, \quad (X_0^i)_{i \in \mathbb{N}} = \mathbf{s} = (s_i)_{i \in \mathbb{N}}$$

on the same Brownian motion $\mathbf{B} = (B_t^i)_{i \in \mathbb{N}}$. Let

$$X_t = \sum_{i=1}^{\infty} \delta_{X_t^i} \quad \text{and} \quad \widehat{X}_t = \sum_{i=1}^{\infty} \delta_{\widehat{X}_t^i}.$$

Suppose, for μ -a.s. ξ ,

$$P_{\mu_\xi} \circ X_t^{-1} \prec \mu_\xi \quad \text{and} \quad P_{\mu_\xi} \circ \widehat{X}_t^{-1} \prec \mu_\xi \quad (\forall t).$$

Then

$$P_{\mathbf{s}}(\mathbf{X} = \widehat{\mathbf{X}}) = 1 \quad \text{for } \mu\text{-a.s. } \mathbf{s} = \sum_{i=1}^{\infty} \delta_{s_i}$$

.

Uniqueness of strong solutions

1:strongx

Thm 6 (O.-Tanemura). Assume (A1)–(A7). Here (A7) μ is tail trivial.

Then the strong solution $\mathbf{X} = (X^i)$ such that

$$P_\mu \circ X_t^{-1} \prec \mu \quad \text{for all } t$$

is unique for μ -a.e. $x = \sum_i \delta_{x_i}$.

Here X is the unlabeled dynamics of \mathbf{X} :

$$X_t = \sum_i^\infty \delta_{X_t^i}$$

Cor If μ is a determinantal RPF, then the associated ISDE has a unique strong solution that is reversible w.r.t. μ .

• Tail σ -fields of Airy, Sine, Ginibre RPFs with $\beta = 2$ and all other determinantal RPFs are trivial.

Uniqueness of Dirichlet forms

Let $\mathcal{D}_{\text{poly}}^\mu$ be the closure of the set of polynomials on S such that $\mathcal{E}_1^\mu(f, f) < \infty$. Then

$$\mathcal{D}_{\text{poly}}^\mu \subset \mathcal{D}^\mu$$

because polynomials are local and smooth.

Thm 7 (O.-Tanemura '14). Assume (A1)–(A7). Then quasi-regular Dirichlet forms that are extension of $(\mathcal{E}^\mu, \mathcal{D}_{\text{poly}}^\mu)$ are unique.

In particular, $\mathcal{D}_{\text{poly}}^\mu = \mathcal{D}^\mu$, and Lang's construction and O.'s construction are same.

r:df

Remark 1. (1) Dirichlet forms here are same as those constructed by Albeverio-et al, and Yoshida.

(2) If (A5) (non-explosion) does not hold. Then Thm 7 does not hold. This is very natural theorem that says the uniqueness of Dirichlet forms is related to the non-explosion problem of tagged problem.

Idea of "strong sol of ISDEs"

- General theory to construct unique, strong solutions of infinite-dimensional stochastic differential equations
- Weak solution: (O. JPSJ 10, PTRF 12, AOP 13, SPA 13)
- logarithmic derivative d^μ : Very informally,

$$d^\mu(x, y) = \nabla_x \log \mu^{[1]}$$

Here $\mu^{[1]}$ is a 1-Campbell measure of μ .

- μ is quasi-Gibbs with upper semi-continuous potential Ψ .
- marginal assumptions

Then ISDE has a weak solution (\mathbf{X}, \mathbf{B}) :

$$dX_t^i = dB_t^i + \frac{1}{2} d^\mu(X_t^i, \sum_{j \neq i}^{\infty} \delta_{X_t^j}) dt \quad (i \in \mathbb{N})$$

Strong solutions of ISDE: Non Markov type

- Strong solutions and uniqueness:
- We lift weak solutions to strong solutions.
- IFC solutions.
- Tail analysis.
- The key idea is the following:
 - We interpret single ISDE as an infinite system of finite dimensional SDEs with consistency (IFC).
 - We regard the tail σ -field of the labeled path spaces as boundary condition of ISDEs.

Strong solutions of ISDE: Non Markov type

- We consider non-Markov SDEs because the argument is general.

$S = \mathbb{R}^d, [0, \infty), \mathbb{C}, \text{ e.t.c..}$ (the space where particles move),

$W(S^{\mathbb{N}}) = C([0, T]; S^{\mathbb{N}}), (0 < T < \infty)$ (labeled path spaces)

- a quadruplet $(W_{\text{sol}}, \mathbf{S}_0, \{\sigma^i\}, \{b^i\})$

W_{sol} : a Borel subset of $W(S^{\mathbb{N}})$ (space of solutions of ISDE)

\mathbf{S}_0 : a Borel subset of $S^{\mathbb{N}}$ (initial starting points of ISDE)

$\sigma^i, b^i: W_{\text{sol}} \rightarrow W(S^{\mathbb{N}})$ (coefficients of ISDE)

- the ISDE on $S^{\mathbb{N}}$ of the form

$$dX_t^i = \sigma^i(\mathbf{X})_t dB_t^i + b^i(\mathbf{X})_t dt \quad (i \in \mathbb{N}) \quad (5)$$

$$\mathbf{X}_0 = \mathbf{s} = (s_i)_{i \in \mathbb{N}} \in \mathbf{S}_0 \quad (6)$$

$$\mathbf{X} \in W_{\text{sol}}. \quad (7)$$

- $\mathbf{X} = \{(X_t^i)_{i \in \mathbb{N}}\}_{t \in [0, T]}$
- $\mathbf{B} = (B_t^i)_{i \in \mathbb{N}}$ ($i \in \mathbb{N}$) is the $S^{\mathbb{N}}$ -valued standard Brownian motion.

Strong solutions of ISDE: Assump (P1)

$$dX_t^i = \sigma^i(\mathbf{X})_t dB_t^i + b^i(\mathbf{X})_t dt \quad (i \in \mathbb{N})$$

$$\mathbf{X}_0 = \mathbf{s} = (s_i)_{i \in \mathbb{N}} \in \mathbf{S}_0$$

$$\mathbf{X} \in W_{\text{sol}}.$$

(P1) ISDE (5) has a weak solution (\mathbf{X}, \mathbf{B}) . (not a strong solution!)

Here $\mathbf{B} = (B^i)_{i \in \mathbb{N}}$ is Brownian motion on $S^{\mathbb{N}}$

Problem: Prove that \mathbf{X} is a functional of the Brownian motion \mathbf{B}

Idea:

Strong solutions of Infinite-dimensional SDE

\Leftrightarrow

Infinite-many, Finite-dimensional SDEs with Consistency (IFC)

+

Tail Triviality of Labeled path spaces w.r.t. the label (TTL)

Assump (P2) infinite-many, finite-dimensional SDEs with consistency

- \bar{P}_s : a prob meas on $W(S^{\mathbb{N}}) \times W^0(S^{\mathbb{N}})$: dist. of weak sol (\mathbf{X}, \mathbf{B}) .
- $\mathbf{P}_s = \bar{P}_s(\mathbf{X} \in \cdot)$
- $P_{\mathbf{B}r}^{\infty} = \bar{P}_s(\mathbf{B} \in \cdot)$.
- For each $\mathbf{X} \in W_{\text{sol}}$, $s \in \mathbf{S}_0$, and $m \in \mathbb{N}$, we define the new SDE of

$$\mathbf{Y}^m = (Y_t^{m,1}, \dots, Y_t^{m,m})$$

such that

$$\begin{aligned} dY_t^{m,i} &= \sigma^i(\mathbf{Y}^m, \mathbf{X}^{m*})_t dB_t^i + b^i(\mathbf{Y}^m, \mathbf{X}^{m*})_t dt && \text{:q1b} \\ \mathbf{Y}_0^m &= (s_1, \dots, s_m) \in S^m, \quad \text{where } s = (s_i)_{i=1}^{\infty}, && (8) \\ (\mathbf{Y}^m, \mathbf{X}^{m*}) &\in W_{\text{sol}}. \end{aligned}$$

Here we set

$$\begin{aligned} \mathbf{X}^{m*} &= (X_t^{m+1}, X_t^{m+2}, \dots) \\ (\mathbf{Y}^m, \mathbf{X}^{m*}) &= (Y_t^{m,1}, \dots, Y_t^{m,m}, X_t^{m+1}, X_t^{m+2}, \dots). \end{aligned}$$

\mathbf{X}^{m*} is interpreted as a part of the coefficients of the SDE (8). :q1b

In fact, we regard (8) as SDEs of \mathbf{Y} such as :q1b

For $m = 1$

$$dY_t^{1,1} = \sigma^1(\mathbf{Y}^1, \mathbf{X}^{1*})_t dB_t^1 + b^1(\mathbf{Y}^1, \mathbf{X}^{1*})_t dt. \quad (m = 1)$$

For $m = 2$

$$dY_t^{2,1} = \sigma^1(\mathbf{Y}^2, \mathbf{X}^{2*})_t dB_t^1 + b^1(\mathbf{Y}^2, \mathbf{X}^{2*})_t dt \quad (m = 2)$$

$$dY_t^{2,2} = \sigma^2(\mathbf{Y}^2, \mathbf{X}^{2*})_t dB_t^2 + b^2(\mathbf{Y}^2, \mathbf{X}^{2*})_t dt.$$

For $m = 3$

$$dY_t^{3,1} = \sigma^1(\mathbf{Y}^3, \mathbf{X}^{3*})_t dB_t^1 + b^1(\mathbf{Y}^3, \mathbf{X}^{3*})_t dt \quad (m = 3)$$

$$dY_t^{3,2} = \sigma^2(\mathbf{Y}^3, \mathbf{X}^{3*})_t dB_t^2 + b^2(\mathbf{Y}^3, \mathbf{X}^{3*})_t dt$$

$$dY_t^{3,3} = \sigma^3(\mathbf{Y}^3, \mathbf{X}^{3*})_t dB_t^3 + b^3(\mathbf{Y}^3, \mathbf{X}^{3*})_t dt.$$

For $m = 4$

$$dY_t^{4,1} = \sigma^1(Y^4, \mathbf{X}^{4*})_t dB_t^1 + b^1(Y^4, \mathbf{X}^{4*})_t dt \quad (m = 4)$$

$$dY_t^{4,2} = \sigma^2(Y^4, \mathbf{X}^{4*})_t dB_t^2 + b^2(Y^4, \mathbf{X}^{4*})_t dt$$

$$dY_t^{4,3} = \sigma^3(Y^4, \mathbf{X}^{4*})_t dB_t^3 + b^4(Y^4, \mathbf{X}^{4*})_t dt$$

$$dY_t^{4,4} = \sigma^4(Y^4, \mathbf{X}^{4*})_t dB_t^4 + b^4(Y^4, \mathbf{X}^{4*})_t dt.$$

For $m = 5$

$$dY_t^{5,1} = \sigma^1(Y^5, \mathbf{X}^{5*})_t dB_t^1 + b^1(Y^5, \mathbf{X}^{5*})_t dt \quad (m = 5)$$

$$dY_t^{5,2} = \sigma^2(Y^5, \mathbf{X}^{5*})_t dB_t^2 + b^2(Y^5, \mathbf{X}^{5*})_t dt$$

$$dY_t^{5,3} = \sigma^3(Y^5, \mathbf{X}^{5*})_t dB_t^3 + b^5(Y^5, \mathbf{X}^{5*})_t dt$$

$$dY_t^{5,4} = \sigma^4(Y^5, \mathbf{X}^{5*})_t dB_t^4 + b^4(Y^5, \mathbf{X}^{5*})_t dt$$

$$dY_t^{5,5} = \sigma^5(Y^5, \mathbf{X}^{5*})_t dB_t^5 + b^5(Y^5, \mathbf{X}^{5*})_t dt.$$

• • •

Strong solutions of ISDE: (P2) seq of finite-dim SDEs with consistency

$$\begin{aligned} dY_t^{m,i} &= \sigma^i(\mathbf{Y}^m, \mathbf{X}^{m*})_t dB_t^i + b^i(\mathbf{Y}^m, \mathbf{X}^{m*})_t dt & (8) \\ \mathbf{Y}_0^m &= (s_1, \dots, s_m) \in S^m, \\ (\mathbf{Y}^m, \mathbf{X}^{m*}) &\in W_{\text{sol}}. \end{aligned}$$

(P2) The SDE (8) has a unique, strong solution for each $s \in S_0$, $\mathbf{X} \in W_{\text{sol}}^s$, and $m \in \mathbb{N}$.

- (P2) is a reasonable assumption. Since $\mathbf{X} \in W_{\text{sol}}$ and W_{sol} is a nice subset of W , we can assume this for the weak solution we have constructed. Here we use Dirichlet form theory again to prove \mathbf{X} stay W_{sol} .
- In the case of Dyson Brownian motions with $m = 1$, we see that

$$b^1(x, y) = \frac{1}{2} d^\mu(x, y) = \lim_{r \rightarrow \infty} \sum_{j=1, |y_j| < r}^{\infty} \frac{1}{x - y_j}.$$

Hence we see that $b^1(x, \mathbf{X}_t^{1*})$ is locally Lipschitz continuous in x for fixed weak solution \mathbf{X} .

Strong solutions of ISDE: (P3) Tail triviality

Let $\mathcal{T}_{\text{path}}(S^{\mathbb{N}})$ be the tail σ -field of $W(S^{\mathbb{N}})$; we set

$$\mathcal{T}_{\text{path}}(S^{\mathbb{N}}) = \bigcap_{m=1}^{\infty} \sigma[\mathbf{X}^{m*}]. \quad (9) \quad \text{:q0y}$$

(P3) $\mathcal{T}_{\text{path}}(S^{\mathbb{N}})$ is \mathbf{P}_s -trivial for each $s \in \mathbf{S}_0$.

Strong solutions of ISDE: Theorem 8

(P1) ISDE (5) has a solution (\mathbf{X}, \mathbf{B}) .

(P2) SDE (8) has a unique, strong solution for all s, \mathbf{X}, m .

(P3) $\mathcal{T}_{\text{path}}(S^{\mathbb{N}})$ is \mathbf{P}_s -trivial for each $s \in \mathbf{S}_0$.

Thm 8. Assume (P1)–(P3). Then

(1) ISDE (5)–(7) has a strong solution for each $s \in \mathbf{S}_0$.

(2) Let \mathbf{Y}_s and \mathbf{Y}'_s be strong solutions of ISDE (5)–(7) starting at $s \in \mathbf{S}_0$ defined on the same space of Brownian motions \mathbf{B} . Then

$$\mathbf{Y}_s = \mathbf{Y}'_s \text{ a.s.}$$

if and only if

$$\mathbf{Y}_s | \mathcal{T}_{\text{path}}(S^{\mathbb{N}}) = \mathbf{Y}'_s | \mathcal{T}_{\text{path}}(S^{\mathbb{N}}) \quad \text{in Law.} \quad (10)$$

• Since \mathbf{Y}_s is strong solution, $\mathcal{T}_{\text{path}}(S^{\mathbb{N}})$ is trivial w.r.t. \mathbf{Y}_s . Hence (10) is equivalent to

$$\text{Tail}^{[1]}(\mathbf{Y}_s) = \text{Tail}^{[1]}(\mathbf{Y}'_s)$$

Here

$$\text{Tail}^{[1]}(\mathbf{Y}_s) = \{\mathbf{A} \in \mathcal{T}_{\text{path}}(S^{\mathbb{N}}); P(\mathbf{Y}_s \in \mathbf{A}) = 1\}. \quad (11)$$

Strong solutions of ISDE: Idea of Theorem 8 ^{1:q1}

- (P1) ISDE (5) ^{:q0a} has a solution (\mathbf{X}, \mathbf{B}) .
- (P2) SDE (8) ^{:q1b} has a unique, strong solution for all s, \mathbf{X}, m .
- (P3) $\mathcal{T}_{\text{path}}(S^{\mathbb{N}})$ is \mathbf{P}_s -trivial for each $s \in \mathbf{S}_0$.

- (\mathbf{X}, \mathbf{B}) : sol of ISDE by (P1). Let (\mathbf{X}, \mathbf{B}) be fixed. ^{:q1b}
- \mathbf{Y}^m is a unique strong sol of SDE(8) by (P2)
- \mathbf{Y}^m is $\sigma[\mathbf{B}] \vee \sigma[\mathbf{X}^{m*}]$ -m'ble. $\mathbf{X}^{m*} = (X^n)_{m < n < \infty}$.
- $\mathbf{Y}^m = (X^1, \dots, X^m)$. by (P2)
- \mathbf{X} is $\sigma[\mathbf{B}] \vee \mathcal{T}_{\text{path}}(S^{\mathbb{N}})$ -m'ble by $m \rightarrow \infty$.
- $\mathcal{T}_{\text{path}}(S^{\mathbb{N}})$ is trivial by (P3) $\Rightarrow \mathbf{X}$ is a strong solution.

IFC and ISDE

Let (\mathbf{X}, \mathbf{B}) be a weak solution.

- For (\mathbf{X}, \mathbf{B}) we introduce **IFC**: ($m \in \mathbb{N}$)

$$dY_t^{m,i} = \sigma^i(\mathbf{Y}^m, \mathbf{X}^{m*})_t dB_t^i + b^i(\mathbf{Y}^m, \mathbf{X}^{m*})_t dt \quad (8) \quad \text{:q1b}$$

$$\mathbf{Y}_0^m = (s_1, \dots, s_m) \in S^m,$$

$$(\mathbf{Y}^m, \mathbf{X}^{m*}) \in W_{\text{sol}}.$$

- We emphasize IFC is a collection of infinitely many finite dimensional equations:
- This system is equivalent to a single **ISDE**:

$$dX_t^i = \sigma^i(\mathbf{X})_t dB_t^i + b^i(\mathbf{X})_t dt \quad (i \in \mathbb{N}) \quad (5) \quad \text{:q0a}$$

$$\mathbf{X}_0 = \mathbf{s} = (s_i)_{i \in \mathbb{N}} \in \mathbf{S}_0 \quad (6) \quad \text{:q0b}$$

$$\mathbf{X} \in W_{\text{sol}}. \quad (7) \quad \text{:q0c}$$

- We show that these two are equivalent, and by using IFC we give a notion of strong solution and pathwise uniqueness in terms of tail σ -field of the labeled path space.

IFC solutions

- We set

$$\mathcal{S} = W_{\text{sol}} \times W(S^{\mathbb{N}}).$$

- Let $F^m : \mathcal{S} \rightarrow W_{\text{sol}}$ the map defined as

$$F^m(\mathbf{s}, \mathbf{X}, \mathbf{B}) = (\mathbf{Y}^m, \mathbf{X}^{m*}).$$

Here \mathbf{Y}^m is a unique strong solution of (8). (by (P2)).

- Let $\bar{P}_{\mathbf{s}}$ be a probability on \mathcal{S} such that $\bar{P}_{\mathbf{s}}(\mathbf{X}_0 = \mathbf{s}) = 1$. We say

$$\lim_{m \rightarrow \infty} F^m = F^\infty \quad \text{in } W_{\text{sol}} \text{ under } \bar{P}_{\mathbf{s}}$$

if for $\bar{P}_{\mathbf{s}}$ -a.s. \mathbf{s} and $\forall i \in \mathbb{N}$

$$\lim_{m \rightarrow \infty} F^m(\mathbf{s}, \mathbf{X}, \mathbf{B}) = F^\infty(\mathbf{s}, \mathbf{X}, \mathbf{B}) \in W_{\text{sol}},$$

$$\lim_{m \rightarrow \infty} \int_0^t \sigma^i(F^m(\mathbf{s}, \mathbf{X}, \mathbf{B}))_u dB_u^i = \int_0^t \sigma^i(F^\infty(\mathbf{s}, \mathbf{X}, \mathbf{B}))_u dB_u^i,$$

$$\lim_{m \rightarrow \infty} \int_0^t b^i(F^m(\mathbf{s}, \mathbf{X}, \mathbf{B}))_u du = \int_0^t b^i(F^\infty(\mathbf{s}, \mathbf{X}, \mathbf{B}))_u du.$$

IFC solutions

1:ifd Let \bar{P}_s be an IFC solution, and

Lem 1. (8) has an IFC solution \bar{P}_s iff (5) has a weak solution (\mathbf{X}, \mathbf{B}) .

Proof. • Set

$$\mathbf{Y}^\infty = F^\infty(s, \mathbf{X}, \mathbf{B}).$$

Then $(\mathbf{Y}^\infty, \mathbf{B})$ under \bar{P}_s is a weak solution.

• Let (\mathbf{X}, \mathbf{B}) be a weak solution with distribution \bar{P}_s . Since \mathbf{X} is a fix point of F^∞ , \bar{P}_s is an IFC solution. \square

IFC solutions

1:ifd Let $\bar{P}_{s,B} = \bar{P}_s(\mathbf{X} \in \cdot | \mathbf{B})$: the regular conditional probability.

Thm 9. (1) (Y, \mathbf{B}) under \bar{P}_s is a strong solution of (5) ^{:q0a} iff $\mathcal{T}_{\text{path}}(S^{\mathbb{N}})$ is $\bar{P}_{s,B}$ -trivial for $P_{\text{Br}}^{\infty} \mathbf{B}$.

(2) Let \mathbf{X}_s and \mathbf{X}'_s be strong solutions defined on the same Brownian motion starting at s . Then

$$\mathbf{X}_s = \mathbf{X}'_s \text{ for a.s. } \iff \mathbf{X}_s |_{\mathcal{T}_{\text{path}}(S^{\mathbb{N}})} = \mathbf{X}'_s |_{\mathcal{T}_{\text{path}}(S^{\mathbb{N}})} \text{ in Law.} \quad (12) \quad \text{:q1dd}$$

Strong solutions of ISDE:

Application of Thm 8^{1:q1} to
interacting Brownian motions.

Application of Thm 8 to interacting Brownian motions.

- We apply Thm 8 to ISDE (1):

$$dX_t^i = dB_t^i + \frac{1}{2}d^\mu(X_t^i, \sum_{j \neq i} \delta_{X_t^j})dt, \quad (X_0^i)_{i \in \mathbb{N}} = s \quad (1)$$

or more generally

$$dX_t^i = \sigma(X_t^i, \sum_{j \neq i} \delta_{X_t^j})dB_t^i + b(X_t^i, \sum_{j \neq i} \delta_{X_t^j})dt. \quad (1)$$

Here $a = \sigma^t \sigma$ and

$$b(x, y) = \frac{1}{2} \{ \nabla a(x, y) + a(x, y) d^\mu(x, y) \} dt \quad (13)$$

$d^\mu(x, y)$ is the logarithmic derivative (informally) defined as

$$\nabla_x \log \mu^{[1]}$$

with 1-Campbel measure $\mu^{[1]}$ of μ :

$$d\mu^{[1]} = \rho^1(x) \mu_x(dy) dx$$

Strong solutions of ISDE:

We check (P1)–(P3) for (1).

(P1) ISDE (5) has a solution (\mathbf{X}, \mathbf{B}) . (by Thm 1)

(P2) SDE (8) has a unique, strong solution for all s, \mathbf{X}, m .

(P3) $\mathcal{T}_{\text{path}}(S^{\mathbb{N}})$ is \mathbf{P}_s -trivial for each $s \in \mathbf{S}_0$.

- How to prove (P2)? $\Rightarrow \nabla_x d^\mu \in \mathcal{D}_{\text{loc}}^{\mu[1]}$
- How to prove (P3)? \Rightarrow Tail Theorems.

Strong solutions of ISDE: How to prove (P3)

- We give a sufficient condition of (P3):
(P3) $\mathcal{T}_{\text{path}}(S^{\mathbb{N}})$ is \mathbf{P}_s -trivial for each $s \in \mathbf{S}_0$.

- The following implies (P3):

(Q1) μ is tail trivial.

(Q2) $P_\mu \circ X_t^{-1} \prec \mu$ for all t .

(Q3) $P_\mu(\cap_{r=1}^{\infty} \{m_r(X) < \infty\}) = 1$.

Here $S_r = \{|x| < r\}$, $X_t = \sum_{i \in \mathbb{N}} \delta_{X_t^i}$, $X^i = \{X_t^i\}$,

$$m_r = \inf\{m \in \mathbb{N}; X^i \in C([0, T]; S_r^c) \text{ for } m < \forall i \in \mathbb{N}\}.$$

1:tail3

Thm 10. Assume (Q1)–(Q3). Then (P3) holds.

- (Q2) is trivial because the unlabeled dynamics is μ -reversible.
- (Q3) is immediate from Lyons-Zheng decomposition.

Out line of the proof of Thm 10 :

- Let $\mathbf{T} = \{t = (t_1, \dots, t_m); t_i \in [0, T], m \in \mathbb{N}\}$.
- $\tilde{\mathcal{T}}_{\text{path}}(S)$ is the cylindrical tail σ -field of the unlabeled path space

$$\tilde{\mathcal{T}}_{\text{path}}(S) = \bigvee_{t \in \mathbf{T}} \bigcap_{r=1}^{\infty} \sigma[\pi_{S_r^c}(X_t)]. \quad :40a \quad (14)$$

- $\tilde{\mathcal{T}}_{\text{path}}(S^{\mathbb{N}})$ is the cylindrical tail σ -field of the labeled path space:

$$\tilde{\mathcal{T}}_{\text{path}}(S^{\mathbb{N}}) = \bigvee_{t \in \mathbf{T}} \bigcap_{n=1}^{\infty} \sigma[\mathbf{X}_t^{n*}]. \quad :40a \quad (15)$$

Here $\mathbf{X}_t^{n*} = (\mathbf{X}_{t_1}^{n*}, \dots, \mathbf{X}_{t_m}^{n*})$.

- We will deduce the triviality of $\mathcal{T}_{\text{path}}(S^{\mathbb{N}})$ from that of S . We will do this along the scheme: $(P_\mu = \int_S P_s d\mu)$.

$$\begin{array}{ccccccccc} \mathcal{T}(S) & \Rightarrow & \tilde{\mathcal{T}}_{\text{path}}(S) & \Rightarrow & \tilde{\mathcal{T}}_{\text{path}}(S^{\mathbb{N}}) & \Rightarrow & \tilde{\mathcal{T}}_{\text{path}}(S^{\mathbb{N}}) & \Rightarrow & \mathcal{T}_{\text{path}}(S^{\mathbb{N}}) \\ P_\mu & & P_\mu & & P_{\mu^\ell} & & P_s \text{ a.s.s} & & P_s \text{ a.s.s} \end{array}$$

Out line of the proof of Thm ^{1:tail3}10 :

Below we assume (Q1): μ is tail trivial.

$$\mathcal{T}(S) \Rightarrow \tilde{\mathcal{T}}_{\text{path}}(S) \Rightarrow \tilde{\mathcal{T}}_{\text{path}}(S^{\mathbb{N}}) \Rightarrow \tilde{\mathcal{T}}_{\text{path}}(S^{\mathbb{N}}) \text{ a.s.} \Rightarrow \mathcal{T}_{\text{path}}(S^{\mathbb{N}}) \text{ a.s.}$$

1:tt0

Lem 2. $\mathcal{T}(S)$ is $P_s(X_t \in \cdot)$ -trivial for μ -a.s.s.

Proof. Since X_t is μ -reversible, $P_\mu(X_t \in \cdot) = \mu$.

Hence by (Q1) we see that $\mathcal{T}(S)$ is $P_\mu(X_t \in \cdot)$ -trivial.

From this we easily obtain Lem 2. □

1:tt1

Lem 3. $\tilde{\mathcal{T}}_{\text{path}}(S)$ is P_μ -trivial and P_s -trivial for μ -a.s.s.

Proof. From Lem 2 and the Markov property of unlabeled dynamics yields Lem 3. □

1:tt1

$$\mathcal{T}(S) \Rightarrow \tilde{\mathcal{T}}_{\text{path}}(S) \Rightarrow \tilde{\mathcal{T}}_{\text{path}}(S^{\mathbb{N}}) \Rightarrow \tilde{\mathcal{T}}_{\text{path}}(S^{\mathbb{N}}) \text{ a.s.} \Rightarrow \mathcal{T}_{\text{path}}(S^{\mathbb{N}}) \text{ a.s.}$$

1:tt2

Lem 4. $\tilde{\mathcal{T}}_{\text{path}}(S^{\mathbb{N}})$ is \mathbb{P}_{μ^ℓ} -trivial.

Proof. For a label ℓ , we have a natural map $\ell_{\text{path}} : W(S) \rightarrow W(S^{\mathbb{N}})$:

$$\ell_{\text{path}}(X) = (X^1, X^2, \dots).$$

For each $\mathbf{A} \in \bigcap_{n=1}^{\infty} \sigma[\mathbf{X}_t^{n*}]$ for some $t \in \mathbb{T}$, we see

$$\begin{aligned} \ell_{\text{path}}^{-1}(\mathbf{A}) &= \bigcap_{r=1}^{\infty} [\ell_{\text{path}}^{-1}(\mathbf{A}) \cap \{m_r(X) < \infty\}] \\ &\subset \bigcap_{r=1}^{\infty} \sigma[\pi_{S_r^c}(X_t)] \cap \{m_r(X) < \infty\} \\ &= \bigcap_{r=1}^{\infty} \sigma[\pi_{S_r^c}(X_t)] \subset \tilde{\mathcal{T}}_{\text{path}}(S) \end{aligned}$$

□

Out line of the proof of Thm ^{1:tail3}10 :

$$\mathcal{T}(S) \Rightarrow \tilde{\mathcal{T}}_{\text{path}}(S) \Rightarrow \tilde{\mathcal{T}}_{\text{path}}(S^{\mathbb{N}}) \Rightarrow \tilde{\mathcal{T}}_{\text{path}}(S^{\mathbb{N}}) \text{ a.s.} \Rightarrow \mathcal{T}_{\text{path}}(S^{\mathbb{N}}) \text{ a.s.}$$

1:tt3

Lem 5. $\tilde{\mathcal{T}}_{\text{path}}(S)$ is \mathbb{P}_S -trivial for μ^ℓ -a.s.s.

Proof. Easy. □

Out line of the proof of Thm 10 : ^{l:tail3}

$$\mathcal{T}(S) \Rightarrow \tilde{\mathcal{T}}_{\text{path}}(S) \Rightarrow \tilde{\mathcal{T}}_{\text{path}}(S^{\mathbb{N}}) \Rightarrow \tilde{\mathcal{T}}_{\text{path}}(S^{\mathbb{N}}) \text{ a.s.} \Rightarrow \mathcal{T}_{\text{path}}(S^{\mathbb{N}}) \text{ a.s.}$$

l:tt4

Thm 11. $\mathcal{T}_{\text{path}}(S^{\mathbb{N}})$ is \mathbf{P}_s -trivial for μ^ℓ -a.s. s.

The difficulty is that the σ -field $\mathcal{T}_{\text{path}}(S^{\mathbb{N}})$ is not countably determined because tail fields are not topologically well behaved. But if we restrict the support of F^∞ , then $\mathcal{T}_{\text{path}}(S^{\mathbb{N}})$ is countably determined under \mathbf{P}_s .

- Recall the map

$$F^\infty(s, \mathbf{X}, \mathbf{B}) = \mathbf{Y}$$

given by the IFC solution of ISDE.

- Then \mathbf{X} is a weak solution.
- Hence \mathbf{X} is a fix point:

$$F^\infty(s, \mathbf{X}, \mathbf{B}) = \mathbf{X}.$$

1:tail3
Out line of the proof of Thm 10 :

For a measurable space (U, \mathcal{U}) we call a subset $\mathcal{V} \subset \mathcal{U}$ a determination class of (U, \mathcal{U}) if any two probability measures P and Q on (U, \mathcal{U}) are equal if and only if $P(A) = Q(A)$ for all $A \in \mathcal{V}$.

Lem 6. Let $\mathcal{V} = \{V_n\}_{n \in \mathbb{N}}$ be a countable determination class of (U, \mathcal{U}) . Let m be a probability measure on (U, \mathcal{U}) . Then

$$m(V_n) \in \{0, 1\} \text{ for all } V_n \in \mathcal{V} \Rightarrow m(A) \in \{0, 1\} \text{ for all } A \in \mathcal{U}.$$

Proof. Let $N(1) = \{n \in \mathbb{N}; m(V_n) = 1\}$. If $N(1) = \emptyset$, then m is the zero measure. If $N(1) \neq \emptyset$, then we take

$$V = \left(\bigcap_{n \in N(1)} V_n \right) \cap \left(\bigcap_{n \notin N(1)} V_n^c \right).$$

Clearly, we obtain $m(V) = 1$.

Let $A \in \mathcal{U}$. Suppose that $V \cap A \notin \{\emptyset, V\}$. Then we can not determine the value of $m(V \cap A)$ by the value of $m(V_n)$ ($n \in \mathbb{N}$). This yields contradiction. Hence $V \cap A \in \{\emptyset, V\}$. If $V \cap A = \emptyset$, then $m(A) = 0$. If $V \cap A = V$, then $m(A) \geq m(V) = 1$. \square

Out line of the proof of Thm 10 :

- Since $S^{\mathbb{N}}$ is Polish, \exists a countable dense set $S_0^{\mathbb{N}} = \{s_k\}$. Let

$$\mathcal{U} = \cup_{k=1}^{\infty} \mathcal{A}[U_r(s_k); 0 < r \in \mathbb{Q}, k \in \mathbb{N}]$$

Here $U_r(s)$ is a ball centered at s with radius r , $\mathcal{A}[\cdot]$ denotes the algebra generated by \cdot . Let

$$\mathcal{V} = \cup_{j=1}^{\infty} \{(\mathbf{X}_t)^{-1}(\mathbb{A}); \mathbb{A} \in \mathcal{U}^j, t \in \{\mathbb{Q} \cap [0, T]\}^j\}.$$

1:ifd3 Then \mathcal{V} becomes a countable determination class of $(W(S^{\mathbb{N}}), \mathcal{B}(W(S^{\mathbb{N}})))$.

Lem 7. Then, for each $V \in \mathcal{V}$,

$$F^{\infty}(s, \cdot, \mathbf{B})^{-1}(V) \cap W_{T, \text{fix}} \in \tilde{\mathcal{T}}_{\text{path}}(S^{\mathbb{N}}) \cap W_{T, \text{fix}} \quad \text{for } P_{\mathbf{B}r}^{\infty}\text{-a.s. } \mathbf{B}$$

1:ifc4

Lem 8. For each $\mathbb{A} \in \tilde{\mathcal{T}}_{\text{path}}(S^{\mathbb{N}})$,

$$\bar{P}_{s, \mathbf{B}}(\mathbb{A}) \in \{0, 1\} \quad \text{for } P_{\mathbf{B}r}^{\infty}\text{-a.s. } \mathbf{B}$$

Proof of Thm 11:

Thm 11 $\mathcal{T}_{\text{path}}(S^{\mathbb{N}})$ is \mathbf{P}_s -trivial for μ^ℓ -a.s.s. 1:tt4

Proof. From $F^\infty(s, \mathbf{X}, \mathbf{B}) = \mathbf{X}$, we deduce that, for $P_{\mathbf{B}r}^\infty$ -a.s. \mathbf{B} , :45a

$$\bar{P}_{s,\mathbf{B}} \circ F^\infty(s, \cdot, \mathbf{B})^{-1} = \bar{P}_{s,\mathbf{B}}. \quad (16)$$

1:ifc3 1:ifc4

• From Lem 7, Lem 8, and $F^\infty(s, \cdot, \mathbf{B})(W_{T,\text{fix}}) = 1$, for all $\mathbf{V} \in \mathcal{V}$,

$$\bar{P}_{s,\mathbf{B}} \circ F^\infty(s, \cdot, \mathbf{B})^{-1}(\mathbf{V}) \in \{0, 1\} \quad \text{for } P_{\mathbf{B}r}^\infty\text{-a.s. } \mathbf{B}.$$

Since \mathcal{V} is countable, we deduce that, for $P_{\mathbf{B}r}^\infty$ -a.s. \mathbf{B} , :45b

$$\bar{P}_{s,\mathbf{B}} \circ F^\infty(s, \cdot, \mathbf{B})^{-1}(\mathbf{V}) \in \{0, 1\} \quad \text{for all } \mathbf{V} \in \mathcal{V}. \quad (17)$$

:45b

Since \mathcal{V} is a countable determination class, we obtain from (17) and Lem 6 that :45c

$$\bar{P}_{s,\mathbf{B}} \circ F^\infty(s, \cdot, \mathbf{B})^{-1}(\mathbf{A}) \in \{0, 1\} \quad \text{for all } \mathbf{A} \in \mathcal{B}(W(S^{\mathbb{N}})). \quad (18)$$

Hence we deduce that

$$\bar{P}_{s,\mathbf{B}} \circ F^\infty(s, \cdot, \mathbf{B})^{-1} = \delta_{\mathbf{X}} \text{ for some } \mathbf{X} = \mathbf{X}(s, \mathbf{B}) \in W(S^{\mathbb{N}}). \quad (19)$$

:45a

In particular, \mathbf{X} is a function of (s, \mathbf{B}) . This combined with (16) implies that $\bar{P}_{s,\mathbf{B}} = \delta_{\mathbf{X}(s,\mathbf{B})}$, and $\mathcal{T}_{\text{path}}(S^{\mathbb{N}})$ is \mathbf{P}_s -trivial for μ^ℓ -a.s.. \square

END