STOCHASTIC PDES, REGULARITY STRUCTURES, AND INTERACTING PARTICLE SYSTEM

HENDRIK WEBER

ABSTRACT. In this series of lectures we will report on recent progress in the theory of stochastic PDEs. The main aim is to explain the theory of "Regularity structures" developed recently by M. Hairer in [18]. This theory gives a way to study well-posedness for a class of stochastic PDEs that could not be treated previously. Prominent examples include the KPZ equation as well as the dynamic Φ^4_3 model.

Such equations were treated previously as perturbative expansions. Roughly speaking the theory of regularity structures provides a way to truncate this expansion after finitely many terms and to solve a fixed point problem for the "remainder". The key ingredient is a new notion of "regularity" which is based on the terms of this expansion.

In the last lecture I will also discuss how these ideas can be applied to study scaling limits of interacting particle systems.

1. LECTURE 1

In this introductory lecture we give an overview of the scope of the theory. We discuss two prominent examples, the KPZ equation and the dynamic Φ^4_d model. Then we give a short overview over the behaviour of these examples on small scale and about renormalisation.

1.1. Main Examples:

1.1.1. *The KPZ equation.* The KPZ equation is probably the most prominent example that the theory of regularity structures can be applied to. It is given by

$$\partial_t h(t,x) = \partial_x^2 h(t,x) + \frac{1}{2}(\partial_x h(t,x))^2 + \xi(t,x)\ . \tag{KPZ}$$

Here the spatial variable x is one-dimensional. The term $\xi(t,x)$ is a formal expression denoting space-time white noise. It is a quite irregular random distribution that will be discussed in more detail below.

Equation (KPZ) was introduced in [21] in 1986. It models fluctuations around the growth of a flat surface in $1+1$ dimensions. The individual terms on the right hand side of (KPZ) have an immediate interpretation: $\partial_x^2 h$ models diffusion of particles along the surface and ξ models the deposition of new particles. The term $\frac{1}{2}(\partial_x h(t,x))^2$ is a geometric term

Date: August 28, 2014.

that takes into account the fact that the choice of reference frame is arbitrary and has no physical meaning. We refer to [7] for a more detailed discussion of this.

The KPZ equation has recently received a lot of attention: In particular in [28, 2] an exact formula for the one point distribution of solutions to (KPZ) was derived. This formula is susceptible for asymptotic analysis and it could be shown that on large scales the distributions converge to a Tracy-Widom distribution. This result in turn could spectacularly be confirmed in physical experiments in Japan [29].

More generally much interest is centred around the large scale behaviour of solutions to (KPZ). It is widely believed that this behaviour is universal and shared with many other growth models with qualitatively similar features.

The focus of these lectures is quite different from this. It is centred around the behaviour of (KPZ) and similar equations on small scales. One of the aims is to explain how equation (KPZ) should actually be interpreted. The difficulty we will address comes from the irregularity of the white noise term ξ and how it feeds into the non-linearity in (KPZ).

1.1.2. *The Φ^4_d model.* Our second principle example is the dynamic Φ^4_d model. It is formally given by the stochastic partial differential equation (SPDE)

$$\partial_t \varphi(t,x) = \Delta\varphi(t,x) - \varphi^3 - m\varphi + \xi\,. \tag{Φ^4_d}$$

Here the spatial variable x takes values in a d-dimensional space. We will see below that we can treat this model in the cases $d = 1, 2, 3$ only.

The invariant measure of (Φ^4_d) was studied intensively in the seventies in the context of Quantum Field Theory (see e.g. [14, 6]). Formally this measure is given by

$$\mu(d\varphi) \propto \exp\Big(-\frac{1}{8}(\varphi^4 + 2m\varphi^2)\Big)\,\nu(d\varphi) \tag{1.1}$$

where ν is the distribution of a suitable Gaussian field (e.g. Brownian motion in one dimension and the Gaussian free field in two dimensions - of course, this formula has to be interpreted with caution).

This model does have a natural interpretation in statistical mechanics and behaves in many ways like a ferromagnetic Ising model. For example it was shown in [15] that the two dimensional version of (1.1) has a phase transition, very similar to the one observed for the two-dimensional Ising model. Also the argument used in [15] resembles strongly the classical Peierls argument [26] that shows that the Ising model admits a phase transition. Below in Section 5 we will argue that equation (Φ^4_d) can be obtained as the scaling limit of a dynamical Ising model with long range interaction (at least for $d = 1, 2$).

As for the KPZ equation we will be mostly concerned with the behaviour of solutions to (Φ^4_d) on small scales. We will see that at least in the two and three dimensional these solutions are distribution valued which makes the treatment of non-linear terms highly non-trivial.

1.2. **Scaling.**

1.2.1. *Space time white noise.* We start our discussion by explaining in more detail the behaviour of the space-time white noise term ξ that appears in (KPZ) and (Φ^4_d). Formally it is defined as a centred Gaussian random distribution which satisfies

$$\mathbb{E}\xi(t,x)\xi(t',x') = \delta(t-t')\,\delta^d(x-x')\;, \tag{1.2}$$

where δ^d denotes the d-dimensional Dirac δ distribution. Of course, as it stands (1.3) does not make sense but it can be made rigorous by saying that ξ is a centred Gaussian random distribution in $\mathcal{D}'(\mathbf{R}\times\mathbf{R}^d)$ which has the property that for any smooth test function $\varphi \in \mathcal{D}(\mathbf{R}\times\mathbf{R}^d)$ we have

$$\mathbb{E}(\xi,\varphi)^2 = \int_{\mathbf{R}\times\mathbf{R}^d} \varphi^2(t,x)\,dt\,dx\;. \tag{1.3}$$

Remark 1.1. This formula shows in particular that (ξ,φ) can be defined for all $L^2(\mathbf{R}\times\mathbf{R}^d)$ functions φ as an isometry from $L^2(\mathbf{R}\times\mathbf{R}^d)$ to $L^2(\Omega,\mathcal{F},\mathbb{P})$. In the space independent case, i.e. if ξ only depends on t (1.3) yields

$$\mathbb{E}\big[(\xi,\mathbf{1}_{[0,s]})(\xi,\mathbf{1}_{[0,t]})\big] = \int_{\mathbf{R}} \mathbf{1}_{[0,s]}(r)\,\mathbf{1}_{[0,t]}(r)\,dr = s\wedge t\;,$$

giving a rigorous meaning to the statement that the derivative of Brownian motion is white noise.

Below we will frequently need the scaling behaviour of various distributions on small scale. Given a white noise ξ we can *define* the rescaled distribution which formally corresponds to $\xi(\lambda_t\cdot,\lambda_x\cdot)$ through

$$(\xi_{\lambda_t,\lambda_x},\varphi) := (\xi,\lambda_t^{-1}\lambda_x^{-d}\varphi(\lambda_t^{-1}\cdot,\lambda_x^{-1}\cdot))\,.$$

Plugging this into (1.3) we obtain

$$\begin{aligned}\mathbb{E}(\xi_{\lambda_t,\lambda_x},\varphi)^2 &= \int_{\mathbf{R}\times\mathbf{R}^d} \lambda_t^{-2}\lambda_x^{-2d}\varphi(\lambda_t^{-1}t,\lambda_x^{-1}x)^2\,dt\,dx\\ &= \lambda_t^{-1}\lambda_x^{-d}\int_{\mathbf{R}\times\mathbf{R}^d}\varphi^2(t,x)\,dt\,dx\;.\end{aligned} \tag{1.4}$$

In particular, we can conclude that $\xi_{\lambda_t,\lambda_x} \overset{\text{law}}{=} \lambda_t^{-\frac{1}{2}}\lambda_x^{-\frac{d}{2}}\xi$.

1.2.2. *Scaling for SPDEs - subcriticality.* We want to perform a similar scaling analysis for the SPDEs of interest. For the moment we remain at a somewhat formal level. Rigorous statements will be given later.

We start by the simplest example, the stochastic heat equation

$$\partial_t Z(t,x) = \Delta Z(t,x) + \xi(t,x)\,, \tag{SHE}$$

in d spatial dimensions.

Remark 1.2. In parts of the literature the term stochastic heat equation refers to the equation

$$\partial_t Z(t,x) = \Delta Z(t,x) + Z(t,x)\,\xi(t,x)\,,$$

in one spatial dimension. This is an important model closely related to (KPZ). In these notes, however, stochastic heat equation will always refer to (SHE).

Performing the rescaling $\hat{Z}(t,x) = \lambda^\alpha Z(\lambda^\beta t, \lambda^\gamma x)$ and $\hat{\xi} = \lambda^{\frac{\beta}{2}}\lambda^{\frac{d\gamma}{2}}\xi_{\lambda^\beta,\lambda^\gamma}$ we obtain

$$\partial_t \hat{Z} = \lambda^{\beta-2\gamma}\Delta\hat{Z} + \lambda^{\alpha+\frac{\beta}{2}-\frac{d\gamma}{2}}\hat{\xi}\,. \tag{1.5}$$

As we have seen above $\hat{\xi} \overset{\text{law}}{=} \xi$. So if we disregard all boundary and initial conditions, we expect $\hat{Z} \overset{\text{law}}{=} Z$ as soon as

$$\alpha = \frac{d}{2} - 1\,, \qquad \beta = 2\,, \qquad \text{and} \qquad \gamma = 1\,. \tag{1.6}$$

Non-linear equations like (KPZ) and (Φ^4_d) do in general not satisfy such a perfect scale invariance property. The basic assumption that lies at the heart of the theory developed in [18] is that *on small scales* the non-linear term is much smaller than the white noise. Roughly speaking this allows to treat the non-linear equation as a perturbation of (SHE).

Let us illustrate this for (KPZ). We perform the same scaling as in (1.6) and set $\hat{h}(t,x) = \lambda^{-\frac{1}{2}}h(\lambda^2 t, \lambda x)$. We obtain

$$\partial_t \hat{h} = \partial_x^2\hat{h} + \frac{\lambda^{\frac{1}{2}}}{2}(\partial_x\hat{h})^2 + \hat{\xi}(t,x)\,.$$

On small scales, i.e. for $\lambda \to 0$, the prefactor $\lambda^{\frac{1}{2}}$ of the non-linear term goes to zero.

A similar statement is true for (Φ^4_d). For this discussion the mass term $m\varphi$ is irrelevant and we drop it. Then applying the scaling (1.6), i.e. setting $\hat{\varphi}(t,x) = \lambda^{\frac{d}{2}-1}\varphi(\lambda^2 t, \lambda x)$ we get

$$\partial_t\hat{\varphi}(t,x) = \Delta\hat{\varphi}(t,x) - \lambda^{4-d}\hat{\varphi}^3 + \hat{\xi}\,.$$

If the spatial dimension d is strictly less than 4 the prefactor λ^{4-d} vanishes in the limit $\lambda \to 0$. We will call this the *subcritical* case. If $d = 4$ the prefactor λ^{4-d} is one and we call this the *critical* case. For $d \geq 5$ (Φ^4_d) is *supercritical*.

The following somewhat vague "definition" will be sufficient for our purpose. We refer the reader to [18, Assumption 8.3] for a more precise

definition which also allows for a more general choice of noise term ξ and linear operator.

Definition 1.3. *Consider the equation*

$$\partial_t u = \Delta u + F(u, \nabla u) + \xi \,, \tag{1.7}$$

in d spatial dimensions. Equation (1.7) is called subcritical *if under the scaling (1.6) the non-linear term $F(u, \nabla u)$ gets transformed into a term $F_\lambda(u, \nabla u)$ which formally goes to zero as λ goes to 0.*

The main result of [18] can roughly be paraphrased as follows. Again we stay at a somewhat formal level.

Metha-Theorem 1.4 ([18])**.** *Assume that SPDE (1.7) is subcritical. We assume that x takes values in a compact subset of $\mathbf{R}^d$ with some boundary conditions. Furthermore, we prescribe an initial condition u_0 which has the same spatial regularity as we expect for the solution u.*

There is a natural notion of solution to (1.7) and such solutions exist and are unique on a time interval $[0, T)$ for some random $T > 0$.

Remark 1.5. The assumption of subcriticality is not just a technical restriction. For example, a non-trivial Φ^4_4 theory is not expected to exist [1].

Remark 1.6. When studying (KPZ) on *large* scales the rescaling (1.6) is not appropriate. On large scales the non-linear term dominates the evolution. For *large* values of λ one should apply the famous $1, 2, 3$-scaling $\hat{h}(t, x) = \lambda^{-\frac{1}{2}} h(\lambda^{\frac{3}{2}} t, \lambda x)$. Then, setting $\hat{\xi} = \lambda \xi_{\lambda^{\frac{3}{2}}, \lambda}$ equation (KPZ) turns into

$$\partial_t \hat{h}(t, x) = \frac{1}{2}(\partial_x \hat{h})^2 + \lambda^{-\frac{1}{2}} \partial_x^2 \hat{h} + \lambda^{-\frac{1}{4}} \hat{\xi}(t, x) \,. \tag{1.8}$$

As λ goes to ∞ the solution of (1.8) is conjectured to converge to an object called the *KPZ fixed point* (see [8]). This limiting object is not yet very well understood. It does *not* coincide with the entropy solution of the Hamilton-Jacobi equation $\partial_t h = \frac{1}{2}(\partial_x h)^2$.

1.3. **The need for renormalisation.** Of course, we need to make more precise what is meant by *solution theory* and *uniqueness* in the *Methatheorem* 1.4.

Classical solution theories for SPDEs (see e.g. [10, 17, 27]) do not apply, because solutions are too irregular. We will see below that for a fixed value of t the solution $h(t, x)$ viewed as a function of the spatial variable x is continuous but not differentiable. The mapping $x \mapsto h(t, x)$ is almost surely α-Hölder continuous for every $\alpha < \frac{1}{2}$ but for no $\alpha \geq \frac{1}{2}$. The distributional derivative $\partial_x h$ exists and it follows from the discussion below that for fixed value of t the function $x \mapsto \partial_x h$ behaves like a spatial white noise. So squaring it seems like a scary enterprise.

For (Φ^4_d) the situation is similar: In the one dimensional situation φ is α-Hölder for every $\alpha < \frac{1}{2}$. This is largely sufficient to define φ^3 and indeed the solution theory for this equation is by now well understood (see e.g.[12]). In the cases $d = 2, 3$ the subcriticality assumption stated in Definition 1.3 applies but still solutions have rather poor regularity properties. In both cases solutions will be only exists as distributions and there is a priori no natural interpretation for the non-linear term φ^3.

The most naive approach is regularisation. If we could show that all suitable regularisations of an equation converge and have the same limit, we could simply define this limit as the solution to this equation.

Unfortunately, this naive approach does not work! Let us illustrate this for the dynamic Φ^4_2 model: One natural approximation consists of replacing the space time white noise by a smooth noise. This can for example be done by convolution. Let $\varrho: \mathbf{R} \times \mathbf{R}^d \to \mathbf{R}$ be a smooth function with compact support and $\int \varrho(t,x)\,dt\,dx = 1$. Then for $\delta > 0$ set

$$\varrho_\delta(t,x) = \delta^{-(2+d)}\varrho(\delta^{-2}t, \delta^{-1}x)\,. \tag{1.9}$$

For reasons we will explain below, we will always stick to the parabolic scaling $\delta^{-2}t$ and $\delta^{-1}x$, but on the level of the regularisation this choice is not too important.

Finally, we define the regularised noise ξ_δ through the convolution $\xi_\delta = \xi \star \varrho_\delta$. For any fixed $\delta > 0$ ξ_δ is a smooth function and solving the equation

$$\partial_t \varphi_\delta = \Delta\varphi_\delta - \varphi_\delta^3 + \xi_\delta \tag{1.10}$$

poses no problem in any dimension. In [20] such an approximation was studied on the two dimensional torus $\mathbb{T}^2$. It was shown that as δ goes to 0 the solutions φ_δ converge to the trivial limit 0 for any initial condition!

In order to obtain a non-trivial limit the scheme (1.10) has to be modified. We will see below that if instead of (1.10) we consider

$$\partial_t \varphi_\delta = \Delta\varphi_\delta - (\varphi_\delta^3 - 3C_\delta\varphi_\delta) + \xi_\delta\,, \tag{1.11}$$

for a suitable dimension dependent choice of *renormalisation* constant C_δ, then in dimensions $d = 2, 3$ the solutions φ_δ do indeed converge to a non-trivial limit φ. In dimension $d = 2$, one can take $C_1 \log(\delta^{-1})$ for a suitable constant C_1, while in the three dimensional case one has to take $C_\delta = C_1\delta^{-1} + C_2 \log(\delta^{-1})$. In particular, in both cases C_δ diverges as δ goes to 0.

A similar *renormalisation* procedure is necessary for the KPZ equation. In [19] it was shown that solutions of

$$\partial_t h_\delta(t,x) = \partial_x^2 h_\delta(t,x) + \frac{1}{2}(\partial_x h_\delta(t,x))^2 - C_\delta + \xi_\delta(t,x) \tag{1.12}$$

on the one-dimensional torus converge to a non-trivial limit h if we choose $C_\delta = C_1\delta^{-1}$ for a suitable constant C_1 (that depends on the choice of mollifier ϱ). We will call these limits φ and h *solutions* to (Φ^4_d) and (KPZ). These limits are unique up to the choice of a single parameter. Indeed, if we change any of the renormalisations C_δ by a constant independent of δ the schemes (1.11) and (1.12) converge to a different limit. The choice of renormalisation constant is *canonical* only up to the choice of this constant.

The theory developed in [18] gives a "recipe" to device the right renormalisation and to prove such convergence results for all subcritical equations. Furthermore, it allows a detailed analysis of the small scale behaviour of the solutions obtained in this way. These "solutions" are in general unique up to the choice of finitely many constants which can explicitly be characterised.

At first sight one might think that by adding constants in the schemes (1.11) and (1.12) we have changed the equation and that we might have lost the physical interpretation by doing this. Remarkably this turns out not to be the case! There is strong evidence - at least for KPZ and for Φ^4_d that the *renormalised* solutions are the physical solutions. For the KPZ equation subtracting a diverging constant corresponds simply to a change of reference frame. Furthermore, it was shown in [4] that the solutions to KPZ arise as scaling limits for the weakly asymmetric simple exclusion process, a natural surface growth model. We will see below that the dynamic φ^4_2 model can also be obtained as a scaling limit of a particle model, namely a near critical Ising model with long range interaction near criticality. As we will see there the "infinite" renormalisation constant has a natural interpretation as shift of the critical temperature.

1.4. **Regularity.** To end this introductory lecture we want to briefly discuss how to measure the regularity of distributions in general and of white noise in particular. For functions of "positive" regularity we will always take the usual Hölder spaces $\mathcal{C}^\alpha$. Below we will always deal with compact domains in space time there is no need to distinguish between the space of locally and globally Hölder functions. Furthermore, from a viewpoint of regularity theory the spaces for integer values of α are not very nice and we will always assume that $\alpha \notin \mathbf{Z}$.

Before we give a definition of spaces of negative regularity, we want to introduce one more twist. As we are dealing with a parabolic operator and much of the analysis revolves around scaling it is useful to measure time and space regularity in a different way. Corresponding to the scaling $(t,x) \mapsto (\lambda^2 t, \lambda x)$ we will say that a function $u\colon \mathbf{R}\times\mathbf{R}^d$ is $\mathcal{C}^\alpha_{\mathfrak{s}}$ for $\alpha \in (0,1)$ if

$$\frac{u(t,x) - u(t',x')}{|t-t'|^{2\alpha} + |x-x'|^\alpha}$$

is (locally) uniformly bounded over all $(t,x) \neq (t',x')$ and a similar definition holds for general $\alpha > 0$ (the subscript $\mathfrak{s}$ in $\mathcal{C}^\alpha_\mathfrak{s}$ represents the choice of parabolic scaling).

The corresponding notion of spaces of negative regularity $\alpha < 0$ is given by the Besov spaces $\mathcal{B}^\alpha_{\infty,\infty}$ (with a similar convention on the space-time scaling). In analogy to the positive Hölder spaces we simply denote this space by $\mathcal{C}^\alpha_\mathfrak{s}$. There are several ways to characterise these spaces, including Paley-Littlewood decomposition ([3]) or a more general wavelet decompositions. For our purposes the following definition is convenient.

Definition 1.7. *Let $\psi \in \mathcal{C}^r(\mathbf{R}\times\mathbf{R}^d)'$ for some r large enough be a distribution. We say that ψ is in $\mathcal{C}^\alpha_\mathfrak{s}$ for $-r < \alpha < 0$ if for all compact sets $\mathfrak{K} \subseteq \mathbf{R}\times\mathbf{R}^d$ there exists a $C_\mathfrak{K} < \infty$ such that for all $z = (t,x) \in \mathbf{R}\times\mathbf{R}^d$, all $0 < \lambda < 1$ and for all test functions $\eta \in \mathcal{C}^r(\mathbf{R}^d)$ with $\|\eta\|_{\mathcal{C}^r} \le 1$ and support contained in the unit ball $B(1,0) \subseteq \mathbf{R}\times\mathbf{R}^d$ we have*

$$\langle \psi, \mathcal{S}^\lambda_z \eta\rangle \le C\lambda^\alpha \,. \tag{1.13}$$

Here we have set

$$\mathcal{S}^\lambda_z \eta(s,y) = \lambda^{-d-2}\,\eta\big(\lambda^{-2}(s-t), \lambda^{-1}(y-x)\big)\,.$$

The smallest constant C that satisfies (1.13) will be called $\|\psi\|_{\mathcal{C}^\alpha_\mathfrak{s}(\mathfrak{K})}$.

Remark 1.8. It is easy to check that $\mathcal{C}^\alpha_\mathfrak{s}$ functions with $\alpha \in (0,1)$ satisfy a similar condition to (1.13). If a function $u\colon \mathbf{R}\times\mathbf{R}^d \to \mathbf{R}$ is in $\mathcal{C}^\alpha_\mathfrak{s}$ for $\alpha \in (0,1)$ then for η as above

$$\langle u(\cdot) - u(z), \mathcal{S}^\lambda_z \eta\rangle \le C\lambda^\alpha \,. \tag{1.14}$$

The regularity of the space time-white noise is now easy to guess. A calculation similar to (1.4) shows that

$$\mathbb{E}\langle \xi, \mathcal{S}^\lambda_z \eta\rangle^2 \le C\lambda^{-d-2}\,, \tag{1.15}$$

which suggests that ξ has regularity $\alpha = -\frac{d}{2} - 1$. The following "Kolmogorov like" theorem shows that this is almost true.

Theorem 1.9. *Let $\xi(\eta)$ be a family of random variables indexed by all $\eta \in L^2(\mathbf{R}\times\mathbf{R}^d)$ and let $\alpha < 0$. Assume that there exists a constant C such that for all smooth functions $\eta\colon \mathbf{R}\times\mathbf{R}^d \to [0,1]$ with compact support in $B(1,0)$ we have and for all $z \in \mathbf{R}\times\mathbf{R}^d$*

$$\mathbb{E}\big|\xi(\mathcal{S}^\lambda_z\varphi)\big|^p \le C\lambda^{\alpha p}\,.$$

Then there exists a random distribution $\tilde\xi$ in $\mathcal{D}(\mathbf{R}\times\mathbf{R}^d)$ such that for all φ we have $\xi(\varphi) = \tilde\xi(\varphi)$ almost surely. Furthermore, for any $\alpha' < \alpha - \frac{d}{p}$ and any compact $\mathfrak{K} \subseteq \mathbf{R}\times\mathbf{R}^d$ we have

$$\mathbb{E}\|\tilde\xi\|^p_{\mathcal{C}^{\alpha'}_\mathfrak{s}(\mathfrak{K})} < \infty\,.$$

Combining this theorem with the fact that for Gaussian random variables bounds on the variance such as (1.15) imply equivalent bounds on all stochastic moments we can conclude that indeed ξ is a distribution in $\mathcal{C}_{\mathfrak{s}}^{-\frac{d}{2}-1-\kappa}$ for every $\kappa > 0$.

2. Lecture 2

We start the second lecture with a review of classical solution techniques for semilinear (stochastic) PDEs. Then we discuss how this approach fails for many interesting equations, such as the dynamic Φ^4_2 and Φ^4_3 model. We illustrate how the difficulty can be overcome by a *perturbative* approach and a suitable renormalisation. The aim of this lecture is to show some actual calculations before embarking into the abstract theory in Section 3.

2.1. **Classical Theory.**

2.1.1. *The linear case.* We first recall the classical Duhamel's principle or *variation of constant formula*. Consider the following inhomogeneous heat equation,

$$\partial_t u = \Delta u + f$$
$$u(t,\cdot) = u_0 \,.$$

From now on we will always assume periodic boundary conditions, i.e. the spatial variable x takes values in the d-dimenisonal torus $\mathbb{T}^d$ identified with $[-\pi,\pi]^d$. Under very general regularity assumptions on f and u_0 the solution is given by the formula

$$u(t,x) = \int_0^t \int_{\mathbb{T}^d} K(t-s,x-y)\, f(s,y)\, dy\, ds + \int_{\mathbb{T}^d} K(t,x-y) u_0(y)\,. \quad (2.1)$$

Here K is the heat kernel on the torus, which for $t > 0$ and $x \in \mathbb{T}^d$ given by

$$K(t,x) = \sum_{k \in 2\pi\mathbf{Z}^d} \frac{1}{(4\pi t)^{\frac{d}{2}}} \exp\Big(-\frac{(x-k)^2}{t}\Big)\,.$$

It is convenient to extend K onto all of $\mathbf{R}\times\mathbb{T}^d \setminus \{(0,0)\}$ by defining it as 0 on $((-\infty,0]\times\mathbb{T}^d)\setminus\{(0,0)\}$. Defined in this way K it is smooth outside of the origin.

To keep the exposition simple we will from now on always assume that the initial condition u_0 is zero. Furthermore, we denote the domain of integration in (2.1) by $\Lambda_t := [0,t]\times\mathbb{T}^d$.

It is an important classical result (essentially a version of the parabolic Schauder estimates, see e.g. [22] or [18, Sec. 5]) that the convolution with the heat kernel K improves regularity by 2 in the scaled regularity sense introduced above. More precisely if $f \in \mathcal{C}^\alpha_{\mathfrak{s}}(\mathbf{R}_+\times\mathbb{T}^d)$ for some $\alpha \notin \mathbf{Z}$, then

$$u(t,x) := \int_{\Lambda_t} K(t-s,x-y)\, f(s,y)\, dy\, ds \quad (2.2)$$

takes values in $\mathcal{C}_{\mathfrak{s}}^{\alpha+2}$. This result is precisely the reason for introducing the anisotropic regularity mentioned above.

The same principle can be applied to solve the stochastic heat equation (SHE). Formally the solution (with zero initial condition) is given by the expression

$$Z(t,x) = \int_{\Lambda_t} K(t-s, x-y)\, \xi(s,y)\, dy\, ds\,. \tag{2.3}$$

This expression is often interpreted as a stochastic integral, but in fact once we have established that white noise is a distribution in $\mathcal{C}_{\mathfrak{s}}^{-\frac{d}{2}-1-\kappa}(\mathbf{R}\times\mathbb{T}^d)$ it can also be defined in a deterministic fashion through duality. The "Schauder estimate" mentioned above then implies that Z has regularity (in the scaled sense) $\mathcal{C}_{\mathfrak{s}}^{1-\frac{d}{2}-\kappa}$ for every $\kappa > 0$. In particular, in one spatial dimension Z is a continuous function, whereas for $d \geq 2$ Z is a genuine distribution.

The regularity of Z can also be obtained directly by a calculation based on the scaling of the heat kernel and the calculation is instructive. We start by observing that for $\lambda > 0$ small enough and $x \in \mathbb{T}^2$

$$K(\lambda^2 t, \lambda x) = \sum_{k \in 2\pi\mathbf{Z}^d} \lambda^{-d} \frac{1}{(4\pi t)^{\frac{d}{2}}} \exp\Big(-\frac{|x - 2\pi\lambda^{-1}k|^2}{t}\Big) \leq \lambda^{-d} \bar{K}(t,x)\,,$$

where $\bar{K}(t,x) = \frac{1}{(2\pi t)^{\frac{d}{2}}} \exp\Big(-\frac{|x|^2}{t}\Big)$. Again we use the convention to extend K to negative times by 0.

Let us calculate the expected fluctuations of Z around a fixed point $\bar{z} \in \mathbf{R}_+ \times \mathbb{T}^d$. For convenience we assume $\bar{z} = (1,0)$. Let η be a smooth function with compact support contained in the unit ball in $\mathbf{R}\times\mathbf{R}^d$.

Then we obtain from (1.3)

$$\mathbb{E}\langle Z, \mathcal{S}^\lambda_{\bar{z}}\eta\rangle^2 = \int_{\Lambda_2} \Big(\int_{\Lambda_2} K(z_1 - z_2)\, \mathcal{S}^\lambda_{\bar{z}}\eta(z_1)\, dz_1\Big)^2 dz_2\,.$$

Here we have combined the space and time variables t and x into a single variable z. The domain of integration is $\Lambda_2 = [0,2]\times\mathbb{T}^2$. A simple rescaling yields

$$\mathbb{E}\langle Z, \mathcal{S}^\lambda_{\bar{z}}\eta\rangle^2 \leq \lambda^{2-d} \int_{\mathcal{S}^\lambda\Lambda_2} \Big(\int_{\mathcal{S}^\lambda\Lambda_2} \bar{K}(z_1 - z_2)\, \eta(z)\, dz_1\Big)^2 dz_2\,. \tag{2.4}$$

where $\mathcal{S}^\lambda\Lambda_2 = [0, 2\lambda^{-2}] \times \lambda^{-1}\mathbb{T}^d$.

In the one dimensional case this calculation does not help. Indeed, in this case the integral appearing on the right hand side of (2.4) diverges linearly in λ^{-1}. This corresponds to the fact that for *positive* regularly $\mathbb{E}\langle Z, \mathcal{S}^\lambda_{\bar{z}}\eta\rangle^2$ is not the right quantity to consider (compare (1.14)). If it is replaced by $\mathbb{E}\langle Z(\cdot) - Z(\bar{z}), \mathcal{S}^\lambda_{\bar{z}}\eta\rangle^2$ we do get the right bound of order λ. For $d = 2$ the integral $\int_{\mathcal{S}^\lambda\Lambda} \Big(\int_{\mathcal{S}^\lambda\Lambda} \bar{K}(z_1 - z_2)\, \eta(z)\, dz_1\Big)^2 dz_2$ diverges logarithmically in λ^{-1}. As above we can use the Gaussianity of Z to turn L^2 bounds into L^p bounds for any p so that Theorem 1.9 implies

that Z attains values in every space $\mathcal{C}_{\mathfrak{s}}^{-\kappa}$ for $\kappa < 0$. For $d \geq 3$ finally, the integrals are uniformly bounded in λ and we get the desired $\mathcal{C}_{\mathfrak{s}}^{\frac{2-d}{2}-\kappa}$ regularity.

2.1.2. *The nonlinear case.* For non-linear equations Duhamel's principle turns into a fixed point problem: We illustrate this for the Φ^4 equation in one spatial dimension. For $d = 1$ equation (Φ^4_d) can be rewritten as

$$\begin{aligned}\varphi(t,x) = &\int_0^t \int_{\mathbb{T}^1} K(t-s,x-y)\,\xi(s,y)\,dy\,ds \\ &- \int_0^t \int_{\mathbb{T}^1} K(t-s,x-y)\,\varphi^3(s,y)\,dy\,ds\,, \end{aligned} \tag{2.5}$$

where as above we have dropped the mass term $m\varphi$ in (Φ^4_d) and set the initial condition to be 0. As we have seen above the stochastic convolution takes values in $\mathcal{C}_{\mathfrak{s}}^{\frac{1}{2}-\kappa}(\mathbf{R}_+ \times \mathbb{T}^1)$ for any κ. Hence, it is natural to solve a fixed point problem in this space. In this regularity class the definition of φ^3 poses no difficulty and it is straightforward to see that for any fixed realisation of Z there exists a (random) T such the operator

$$\begin{aligned}\Phi\colon \varphi \mapsto &\int_0^{\cdot} \int_{\mathbb{T}^1} K(\cdot - s,\cdot - y)\,\xi(s,y)\,dy\,ds \\ &- \int_0^{\cdot} \int_{\mathbb{T}^1} K(\cdot - s,\cdot - y)\,\varphi^3(s,y)\,dy\,ds \end{aligned} \tag{2.6}$$

is a contraction on bounded balls in $\mathcal{C}_{\mathfrak{s}}^{\frac{1}{2}-\kappa}([0,T]\times\mathbb{T}^1)$. Let us point out that the term $v = \varphi - Z = -\int_0^t \int_{\mathbb{T}^1} K(t-s,x-y)\,\varphi^3(s,y)\,dy\,ds$ is much more regular than φ itself. The Schauder estimates mentioned above imply that it is actually a $\mathcal{C}_{\mathfrak{s}}^{\frac{5}{2}-\kappa}$ function.

Remark 2.1. Note that this argument does *not* make use of the sign of the nonlinear term $-\varphi^3$. Of course, this sign is essential when deriving bounds that imply non-explosion, the existence of invariant measures for solutions, or even existence and uniqueness on the full real line.

2.2. **The higher dimensional case.** For $d \geq 2$ it is not so easy to solve the fixed point problem (2.5) (with the one-dimensional torus $\mathbb{T}^1$ replaced by $\mathbb{T}^2$). As we have seen above the stochastic convolution Z only takes values in the distributional spaces $\mathcal{C}_{\mathfrak{s}}^{-\frac{2-d}{2}-\kappa}$. But in this regularity class the mapping $\varphi \mapsto \varphi^3$ is not defined.

We will now perform the Picard iteration step by step and see where exactly the problem arises. More precisely we set $\varphi_0 = 0$ and aim to study the behaviour of the sequence (φ_n) defined recursively as

$$\varphi_{n+1} = \Phi(\varphi_n)\,,$$

where Φ is defined in (2.6). According to our choice $\varphi_0 = 0$ the first step in the Picard iteration yields the stochastic convolution we already discussed above: $\varphi_1 = Z$.

The problem arises when applying Φ to Z, because as mentioned above Z does not have sufficient regularity to define Z^3. Of course, in general there cannot be a consistent way of cubing an arbitrary distribution of regularity $\mathcal{C}_{\mathfrak{s}}^{-\kappa}$ but Z is *not* an arbitrary distribution. Indeed, Z is a Gaussian random variable and we have precise control on its covariance structure.

Let us try to define Z^3 by approximation: We define a smoothing kernel ϱ_δ as above in (1.9) and set

$$Z_\delta(t,x) = Z \star \eta_\delta(t,x) = \int_{\Lambda_t} K \star \varrho_\delta(t-s, x-y)\, \xi(s,y)\, dy\, ds\,. \tag{2.7}$$

For every $\delta > 0$ the random function $Z_\delta(t,x)$ is smooth and we can define $Z_\delta(t,x)^3$ without ambiguity. To analyse the passage $\delta \to 0$ we may now make explicit use of the structure of the white noise ξ. The behaviour of integrals against white noise is well understood (see e.g. [25]). These higher dimensional stochastic integrals have very similar behaviour to *usual* one dimensional stochastic integrals. In particular there is an *Itô-like* formula which in the current context states that

$$\begin{aligned} Z_\delta(y)^3 = &\int \mathcal{W}_\delta^{(3)}(y; z_1, z_2, z_3) \xi(dz_1)\, \xi(dz_2)\, \xi(dz_3) \\ &+ 3 \int \mathcal{W}_\delta^{(1)}(y; z_1)\, \xi(dz_1)\,, \end{aligned}$$

where

$$\mathcal{W}_\delta^{(3)}(y; z_1, z_2, z_3) = K \star \eta_\delta(y - z_1)\, K \star \eta_\delta(y - z_2)\, K \star \eta_\delta(y - z_3)\,,$$

and

$$\mathcal{W}_\delta^{(1)}(y; z_1) = K \star \eta_\delta(y - z_1) \int \big(K \star \eta_\delta(y - z)\big)^2\, dz. \tag{2.8}$$

We use the notation $\int \ldots \xi(dz_1)\, \xi(dz_2)\, \xi(dz_3)$ to denote an iterated stochastic integral (see [25]). The variance of the iterated stochastic integrals can be calculated explicitly as

$$\begin{aligned} &\mathbb{E}\Big(\int \mathcal{W}_\delta^{(3)}(y; z_1, z_2, z_3) \xi(dz_1)\, \xi(dz_2)\, \xi(dz_3) \Big)^2 \\ &\qquad = 3! \int \mathcal{W}_\delta^{(3)}(y, z_1, z_2, z_3)^2\, dz_1\, dz_2\, dz_3\,. \end{aligned}$$

The limiting behaviour of the iterated stochastic integrals can now be analysed in the same way as the behaviour of the linear Gaussian process Z. In particular we can calculate explicitly

$$\begin{aligned} &\mathbb{E}\Big\langle \int \mathcal{W}_\delta^{(3)}(y; z_1, z_2, z_3) \xi(dz_1)\, \xi(dz_2)\, \xi(dz_3), \eta_\lambda \Big\rangle^2 \\ &\qquad = \int \Big(\int \mathcal{W}_\delta^{(3)}(y; z_1, z_2, z_3)\, \eta_\lambda(y)\, dy \Big)^2 dz_1\, dz_2\, dz_3 \le C\big(\log(\lambda^{-1})\big)^3\,, \end{aligned} \tag{2.9}$$

for a constant C that does not depend on δ.

The lower order *Itô correction* $3\int \mathcal{W}_\delta^{(1)}(y;z_1)\,\xi(dz_1)$, however, causes problems. The explicit form (2.8) of the kernel $\mathcal{W}_\delta^{(1)}$ shows that it can be rewritten as $3C_\delta Z$ for

$$C_\delta = \int \big(K \star \eta_\delta(y-z)\big)^2\, dz\ .$$

This constant C_δ diverges logarithmically. The solution to this problem is to simply remove the diverging term $3C_\delta Z$ and study the limiting $\delta \to 0$ for

$$Z_\delta^3 - 3C_\delta Z_\delta = \int \mathcal{W}_\delta^{(3)}(y;z_1,z_2,z_3)\xi(dz_1)\,\xi(dz_2)\,\xi(dz_3)\ .$$

In (2.9) we had already obtained bounds on the variance of this expression. It is another nice property of iterated stochastic integrals (or equivalently random variables in a homogenous Wiener chaos) that as in the case of Gaussian random variables bounds on the variance imply equivalent bounds for all moments. Hence we are in the position to apply Theorem 1.9 and conclude the convergence of $Z_\delta^3 - 3C_\delta Z_\delta$ to a limiting distribution in $\mathcal{C}_{\mathfrak{s}}^{-\kappa}$ (this convergence takes place in every stochastic L^p space). We denote this distribution by $:Z^3:$.

We now have a natural candidate $:Z^3:$ for the cube of Z and we can continue the Picard iteration. At this step it becomes convenient to introduce a graphical notation. We denote the process Z by $\Pi\uparrow$. Here the dot at the top of the graph represents an instance of the space time white noise, and the line below represents one integration against the heat kernel. The reason for putting a Π will become clear in the next section. The process $:Z^3:$ is denoted by $\Pi\Psi$, where again each dot represents an occurrence of white noise and each line represents one integration against a kernel. The fact that they are merged at the bottom corresponds to multiplication. In our Picard iteration we set

$$\varphi_2 = \uparrow - K \star (\Pi\Psi)\ .$$

In the next step of the Picard iteration we would formally get

$$\varphi_3 = \Pi\uparrow - K \star \Big(\Pi\Psi + 3\Pi\vee K * (\Pi\Psi) + 3\Pi\uparrow\big(K * \Pi\Psi\big)^2 + \big(K * \Pi\Psi\big)^2\Big)\,.$$

Fortunately, almost all of these terms are well defined. Indeed, according to the Schauder estimates $K \star \Pi\Psi$ is a function of class $\mathcal{C}_{\mathfrak{s}}^{2-\kappa}$ for any $\kappa > 0$. And this is enough to define most of the products. The only term that causes a problem is the term $\Pi\vee = Z^2$. For this term a similar renormalisation procedure has to be performed. It can be shown in the same way as above that $Z_\delta^2 - C_\delta$ (for the same C_δ) converge to a random distribution $:Z^2:$.

It turns out that these are all the terms that need to be renormalised and that after modifying these first few steps in the Picard iteration can actually be closed. This argument was performed in a very elegant way in the two dimensional situation in [9]. In the higher dimensional case $d = 3$ we need to renormalise more terms, e.g.  , etc. The

remarkable result that was achieved in [18] is to show how such a fixed point equation can be closed after the renormalisation of finitely many terms for many equations.

3. Lecture 3

The aim of this third lecture is two-fold. We start by an general overview of the ingredients necessary to solve an irregular SPDE. After that we embark to explain the main objects of the theory developed in [18], namely *regularity structures*, *models*, and *controlled distributions*.

3.1. **General strategy.** The general strategy for the renormalisation procedure when solving a non-linear stochastic PDE using the theory of regularity structures is represented in Figure 1.

At the bottom left corner input to solution theory the "input" of the solution theory is represented. This consists of the initial condition u_0 which takes values in some space $\mathcal{C}^\alpha$, as well as the realisation of the regularised space-time white noise ξ_δ. Of course we expect ξ_δ to converge to ξ only as a distribution on $\mathcal{C}_{\mathfrak{s}}^{-1-\frac{d}{2}-\kappa}$, but for any $\delta > 0$ ξ_δ is actually smooth. The symbol $\mathscr{F}$ represents the exact "form" of the equation. This is mentioned explicitly as it changes throughout the renormalisation procedure. The arrow at the bottom represents the "conventional" solution operator - ξ_δ is a smooth function and one can simply solve the PDE to obtain a solution.

Unfortunately, this solution map is not continuous in the topology in which we expect ξ_δ to converge to ξ . The idea is now to factorize the solution map into two parts: We first lift the information available to us to a "model" depicted on the upper left corner. In the concrete example, discussed in Section 2, for any fixed choice of $\delta > 0$ the there is a *canonical* model containing ξ_δ but also $Z_\delta = K \star \xi_\delta$ as well as Z_δ^2 and Z_δ^3.

As illustrated in Section 2, the convergence of the ingredients of the model is much easier to analyse than the convergence of the full non-linear system. Using probabilistic techniques (essentially an analysis of the integrals describing the covariances) it can be shown that these terms can be renormalised to converge in a *suitable* topology. The arrow labeled $\mathfrak{R}$ on the upper left corner symbolises this renormalisation procedure.

The main point is the observation that the solution map is actually *continous* in this *suitable topology*. In the figure it is subdivided once more into a solution map of an *enhanced* fixed point problem in a space of *modelled distributions* and the projection with the *reconstruction operator*.

The aim of this section, is to describe this *suitable topology*. In the next Section 4 we give an overview over the ingredients which are necessary to set up and solve the *enhanced* fixed point problem.

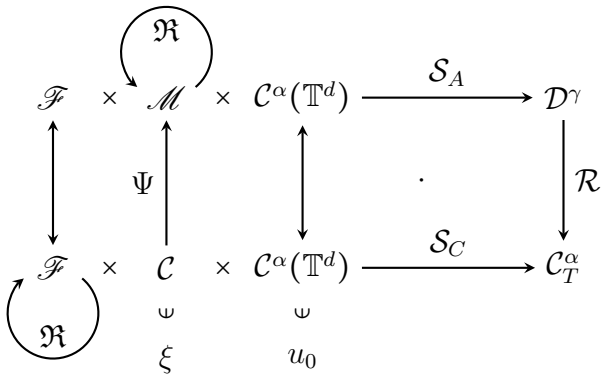

FIGURE 1. The strategy for solving a stochastic PDE.

3.2. **An abstract take on regularity.** The key idea is a new look at the notion of regularity. We begin by a seemingly overcomplicated discussion of γ Hölder functions for $\gamma > 0$, where as above we always assume that $\gamma \notin \mathbf{Z}$. Everything we discuss in this subsection can be carried out in the space-time Hölder spaces with parabolic scaling (and indeed in [18] it is), but for the moment we stick to the isotropic case. The generalisation is straightforward.

Let $\gamma = k + \gamma'$ for $\gamma' \in (0,1)$ and $k \in \mathbf{N}_0$. Usually, a function $f : \mathbf{R}^d \to \mathbf{R}$ is said to be of class $\mathcal{C}^\gamma$ if it is k times continuously differentiable and all derivatives of order k or γ' Hölder.

A nicer way to characterise the same property is to say that locally around every point x the function f can be described by a polynomial up to an error of order γ. More precisely, if f is of class $\mathcal{C}^\gamma$ we have

$$\Big| f(y) - \sum_{|\mathfrak{n}|\le k} \frac{1}{\mathfrak{n}!} D^{\mathfrak{n}} f(x)(y-x)^{\mathfrak{n}} \Big| \le C|y-x|^\gamma \,, \tag{3.1}$$

where the constant can be chosen uniformly over all x and y in any compact set. Here and below we denote by $\mathfrak{n}$ the multi-index $n_1, \dots, n_d$, and we use the conventions $|\mathfrak{n}| = n_1 + \dots n_d$, $\mathfrak{n}! = n_1! \dots n_d!$, $D^{\mathfrak{n}} = \partial_{x_1}^{n_1} \dots \partial_{x_d}^{n_d}$, and finally $(y-x)^{\mathfrak{n}} = (y_1 - x_1)^{n_1} \dots (y_d - x_d)^{n_d}$.

It is useful to add the regularity of the derivatives into the conditions, for every multi index $\mathfrak{n}$ with $|\mathfrak{n}| = \alpha$

$$\Big| D^{\mathfrak{n}} f(y) - \sum_{|\mathfrak{n}+\mathfrak{n}'|\le k} \frac{1}{\mathfrak{n}'!} D^{\mathfrak{n}+\mathfrak{n}'} f(x)(y-x)^{\mathfrak{n}'} \Big) \Big| \le C|y-x|^{\gamma-\alpha} \,. \tag{3.2}$$

We observe that the terms corresponding to the higher orders in the description (corresponding to larger value of $\alpha = |\mathfrak{n}|$) have lower regularity.

This description of $\mathcal{C}^\gamma$ functions can now be put into a – for the moment overly complicated – framework. We denote by $\bar{\mathcal{T}} =$ the space of all abstract polynomials in d variables. We have the natural grading $\bar{\mathcal{T}} = \bigoplus_{\alpha \in \mathbf{N}_0} \mathcal{T}_\alpha$ where $\mathcal{T}_\alpha$ is the space of formal monomials of homogeneity α.

Then any $\mathcal{C}^\gamma$ function $f\colon \mathbf{R}^d \to \mathbf{R}$ can naturally be lifted to a function $F\colon \mathbf{R}^d \to \bar{\mathcal{T}}$ simply setting

$$F(x) = \sum_{|\mathfrak{n}|\le k} D^{\mathfrak{n}} X^{\mathfrak{n}} .$$

Of course, the original function f can be recovered from F by setting

$$f = \mathcal{R}F = F_0 .$$

($F_0(x)$ is the zero order term in $F(x)$).

There exists a natural mapping $\Pi\colon \mathbf{R}^d \times \bar{\mathcal{T}} \to \mathcal{C}^\infty(\mathbf{R}^d)$ which is linear in the $\bar{\mathcal{T}}$ variable and given by $\Pi_x X^{\mathfrak{n}} = (\cdot - x)^{\mathfrak{n}}$. Then for a fixed value of x the function $\Pi_x F(x) = \sum_{|\mathfrak{n}|\le k} D^{\mathfrak{n}}(\cdot - x)^{\mathfrak{n}}$ is a local description of degree γ of f. More precisely

$$\big|\big(\Pi_x F(x)\big)(y) - f(y)\big| \le C|x-y|^\gamma ,$$

locally uniformly in $x \ne y$. Note that this expression is nothing but (3.1)!

We still need to express the conditions (3.1) and (3.2) as a continuity condition on F. To this end we need to express the evaluation of the polynomial $F(x)$ at a different point y on the level of the coefficients. For any monomial $X^{\mathfrak{n}}$ and for $x, y \in \mathbf{R}^d$ we set

$$\Gamma_{x,y} X^{\mathfrak{n}} := \big(X - (y - x)\big)^n .$$

We naturally have $\Pi_y X = \Pi_x \Gamma_{x,y} X$. Then (3.1) and (3.2) can be expressed as Then (3.1) and (3.2) can be written as

$$\sup_{\alpha<\gamma} \frac{|F_\alpha(y) - \Gamma_{xy} F_\alpha(x)|}{|x-y|^{\gamma-\alpha}} \le C .$$

for a constant that can be chosen locally uniformly in $x \ne y$.

The key idea is now to generalise this regularity formalism by allowing additional functions and not only polynomials in the description of the function f. Unlike the polynomials these additional functions will in general not be smooth and sometimes even distributions.

3.3. **Controlled rough paths.** Of course the framework introduced above is much too complicated to only characterise $\mathcal{C}^\gamma$ functions. Gubinelli's theory of *controlled rough paths* (see [16], see [23] for Lyons' original work on rough paths) constitutes a non-trivial example.

Gubinelli studied the classical problem of defining the integral

$$\int_0^1 y \, dx , \tag{3.3}$$

for functions $x, y \in \mathcal{C}^\gamma[0,1]$ for some $\gamma \in (\frac{1}{3}, \frac{1}{2})$. As in the renormalisation procedure discussed above in Section 2, there is no hope to solve this problem in this generality. Gubinelli's (or Lyons') idea was to split the construction of the integral (3.3) into two steps. In a first step the integral $\int x \, dx$ is constructed explicitly (usually by a stochastic

procedure). Then in the second step the integral (3.3) can be constructed for a whole class of functions that are *locally well described by* x. More precisely in [16] defined that a function $y\colon [0,1] \to \mathbf{R}^d$ is *controlled by* a $\mathcal{C}^\gamma$ function if there exists a function $y'\colon [0,1] \to \mathbf{R}^{d\times d}$ such that uniformly for all $s \neq t$

$$\big|y(t) - (y(s) + y'(s)(x(t) - x(s))\big| \leq C|t-s|^{2\gamma}\,, \tag{3.4}$$

and

$$\big|y'(t) - y'(s)\big| \leq C|t-s|^{\gamma}\,. \tag{3.5}$$

We encourage the reader to compare these bounds to the bounds (3.1) and (3.2) above. Gubinelli's observation was that although y itself is only a $\mathcal{C}^\gamma$ function, the bounds (3.1) allow to treat it like a function of $\mathcal{C}^{2\gamma}$ regularity.

These bounds (3.4) and (3.5) can conveniently be put into the same framework discussed above. First of all we set

$$\mathcal{T} = \mathcal{T}_0 \oplus \mathcal{T}_\gamma\,,$$

where $\mathcal{T}_0 = \mathbf{R}^d$ and $\mathcal{T}_\gamma = \mathbf{R}^{d\times d}$. As above we lift the function y to a function $Y\colon [0,1] \to \mathcal{T}$ by setting

$$Y(s) = y(s) \oplus y'(s)\,. \tag{3.6}$$

Again, it is immediate to recover the original function y from Y, by simply setting

$$y = \mathcal{R}Y = Y_0\,,$$

We want to capture the following two facts: On the one hand we want to express that at every point $s \in [0,1]$ we can get a good description of y in a whole neighbourhood of s from the value of $Y(s)$. On the other hand we want to express that Y is regular in a suitable sense. Unlike the case of polynomial, however, we need non-trivial information based on X in order to express this.

For every $s \in [0,1]$ and for $y \oplus y' \in \mathcal{T}$ we define the function

$$\Pi_s(y \oplus y')(t) = y \oplus y'\big(x(t) - x(s)\big)\,.$$

We also define the linear operators $\Gamma_{s,t}\colon \mathcal{T} \to \mathcal{T}$ by

$$\Gamma_{s,t}(y \oplus y') = \big(y + y'(x(t) - x(s)) \oplus y'\big)\,.$$

Now the bounds (3.4) and (3.5) can be re-expressed as

$$\big|\mathcal{R}Y(s) - \Pi_t Y(t)(s)\big| \leq |t-s|^{2\gamma}\,, \tag{3.7}$$

as well as

$$\sup_{\alpha<\gamma} \frac{|Y_\alpha(t) - \Gamma_{st}Y_\alpha(s)|}{|s-t|^{\gamma-\alpha}} \leq C\,. \tag{3.8}$$

Remark 3.1. As they stand the bounds (3.7) and (3.8) seem somewhat redundant. Indeed, the bound for $\alpha = 0$ in(3.8) is equivalent to (3.7). This is because in both examples discussed so far the definition of the *reconstruction operator* $\mathcal{R}$ is trivial. In the examples arising when solving SPDEs this will typically not be the case.

3.4. **Regularity structures.** We are now ready to give an axiomatic framework for the ideas developed in the previous section

Definition 3.2. *A regularity structure consists of a graded vector space $\mathcal{T} = \bigoplus_{\alpha\in A}\mathcal{T}_\alpha$ where A denotes a set of real-valued indices (called homogeneities) that is locally finite and bounded from below. Each of the spaces $\mathcal{T}_\alpha$ is finite-dimensional and comes with a distinguished canonical basis. The space $\mathcal{T}$ also comes endowed with a group $\mathcal{G}$ of continuous linear transformations of $\mathcal{T}$ with the property that, for every $\Gamma \in \mathcal{G}$, every $\alpha \in A$, and every $\tau \in \mathcal{T}_\alpha$ one has*

$$\Gamma\tau - \tau \in \bigoplus_{\beta<\alpha} \mathcal{T}_\beta \,. \tag{3.9}$$

Remark 3.3. It is easy to check that (3.9) is indeed satisfied for polynomials and controlled rough paths.

Definition 3.4. *Let $\mathscr{T} = (A, T, G)$ be a regularity structure. A model for $\mathscr{T}$ on $\mathbf{R}^d$ consists of*

- *A map $\Gamma\colon \mathbf{R}^d \times \mathbf{R}^d \to G$ such that $\Gamma_{xx} = I$, and such that $\Gamma_{xy}\Gamma_{yz} = \Gamma_{xz}$ for all x, y, z.*
- *Continuous linear maps $\Pi_x\colon T \to \mathcal{S}'(\mathbf{R}^d)$ such that $\Pi_y = \Pi_x \circ \Gamma_{xy}$ for all x, y.*

Locally uniformly in x, y we require for $a \in \mathcal{T}_\alpha$

$$|(\Pi_x a)\big(\mathcal{S}_x^\delta \eta\big)| \lesssim \|a\|\delta^\alpha \,, \quad \|\Gamma_{xy}a\|_\beta \lesssim \|a\|\,\|x-y\|^{\alpha-\beta} \,. \tag{3.10}$$

Here $\|\Gamma_{xy}a\|_\beta$ denotes the norm of the component of $\Gamma_{xy}a \in \mathcal{T}_\beta$.

Remark 3.5. The regularity assumption (3.10) is again easily checked for polynomials. For the controlled rough paths they amount to requiring $\mathcal{C}^\gamma$ regularity for X. Note that the first regularity assumption has to be expressed in terms of scaled test functions because in general $\Pi_x a$ is only a distribution. Let us stress furthermore, that we do *not* require $\Pi_x a$ to be a distribution of regularity $\mathcal{C}^\alpha$.

Finally, the generalisation of $\mathcal{C}^\gamma$ is given by.

Definition 3.6. *Fix a regularity structure $\mathscr{T}$ and a model (Π, Γ). Then, for $\gamma \in \mathbf{R}$, the space of modelled distributions $\mathcal{D}^\gamma$ consists of all T_γ^--valued functions f such that*

$$\|f\|_{\gamma;\mathfrak{K}} = \sup_x \sup_{\beta<\gamma} \|f(x)\|_\beta + \sup_{\|x-y\|_{\mathfrak{s}}\le 1} \sup_{\beta<\gamma} \frac{\|f(x) - \Gamma_{xy}f(y)\|_\beta}{\|x-y\|_{\mathfrak{s}}^{\gamma-\beta}} < \infty \,.$$

Here the supremum is taken over all x in the compact set $\mathfrak{K}$.

4. Lecture 4

In this section we first discuss some concrete regularity structures used to solve SPDEs. In a second step we very briefly summarise the operations that can be defined on modelled distributions, and that are then used to set up and solve the *enhanced* fixed point problem.

4.1. **Regularity structures for some SPDEs.**

Example 4.1. We start with the example already mentioned above, the two dimensional Φ^4 theory. As we have already mentioned above, the renormalisation for this equation was already resolved in [9] without regularity structures.

Recall, that the equation is given by

$$\partial_t \varphi(t,x) = \Delta\varphi(t,x) - \varphi^3 + \xi \,. \tag{4.1}$$

As above we have dropped the mass term $m\varphi$. Recall that in the two dimensional situation the solution of the linearised problem (SHE) has regularity $\mathcal{C}^{-\kappa}$ for any $\kappa > 0$ and that we expect φ to have the same regularity. Furthermore, recall that we cannot define the non-linear term φ^3 in this regularity class. But actually, φ only just falls short of the required regularity and this is reflected in the fact that we only need a rather "small" regularity structure.

To solve (4.1) it is sufficient to take

$$\mathcal{T} = \bigoplus_{\tau} \mathbf{R}\tau$$

where the symbol τ takes values in the set

$$\tau \in \{\mathbf{1}, \Irightarrow, \Vee, \Psi\} \,.$$

The interpretation of the graphs is the same as above in Section 2. We need to assign the order α to each of the symbols and we set

$$|\mathbf{1}| = 0 \qquad |\Irightarrow| = -\kappa \qquad |\Vee| = -2\kappa \qquad |\Psi| = -3\kappa \,,$$

for some arbitrary (small enough) value of $\kappa > 0$. The action of the group Γ is completely trivial in this example - $\mathcal{G}$ only consists of the identity.

The construction of the model in this case was already describe in Section 2. Indeed we simply take

$$\Pi_z \mathbf{1} = 1 \qquad \Pi_z \Irightarrow = \Pi \Irightarrow = Z \qquad \Pi_z \Vee = \Pi \Vee =: Z^2 : \qquad \Pi_z \Psi = \Pi \Psi =: Z^3 : \,.$$

Let us stress that – unlike the general case – the model does *not* depend on the base point z at all.

Given this model (which of course has to be constructed in a probabilistic fashion) we construct the solution U to the *enhanced* fixed point problem by a completely deterministic method. Let us stress that in order to give a local description of U itself we do not need the full regularity structure. Indeed, U only takes values in a *sector* (which essentially means sub-regularity structure) of $\mathcal{T}$. The solution U maps

into $\mathbf{R}\mathbf{1} \oplus \mathbf{R}\uparrow$. The remaining symbols $\curlyvee$ and Ψ are only necessary for the construction of the fixed point map. We will sometimes write $U \in \mathcal{D}^{\gamma}_{-\kappa}$ $\gamma > 2\kappa$ to encode the fact that U only has a non-trivial contribution to spaces of order $\geq \alpha$.

Remark 4.2. We would be free to take a larger regularity structure $\mathcal{T}$ to obtain a better local description of the solution to (4.1). The choice presented here is the *minimal* choice necessary to define the fixed point map. In principle it would be possible to get a description of $U \in \mathcal{D}^{\gamma}_{\alpha}$ for an arbitrary γ, but in practice this is always restricted by the impact of the initial condition which we ignored.

Example 4.3. The regularity structure necessary for the renormalisation of (4.1) in three spatial dimensions contains many more terms. The list of terms which are needed in the description of U is again obtained by iteratively going through the Picard iteration and collecting terms. We need to keep all the terms of order ≤ 1. In this way the minimal regularity structure is given by

$$\mathcal{T} = \bigoplus_{\tau} \mathbf{R}\tau$$

where

$$\tau \in \{\Xi, \mathbf{1}, \uparrow, \curlyvee, \Psi, \Psi\curlyvee, \Psi\Psi, \Psi\curlyvee\curlyvee\} \cup \{X, \uparrow X, \curlyvee X, \curlyvee X^2, \Psi X, \Psi X^2, \mathcal{I}\curlyvee, \mathcal{I}\Psi\} \ . \quad (4.2)$$

Here, X_i is an abstract expression that should be replaced by all formal monomials X_i $i = 1, 2, 3$, and X^2 should be replaced by all $X_i X_j$ for $i, j = 1, 2, 3$ or by X_0 which represents the time variable. $\mathcal{I}$ is a formal representation of the integration against the heat kernel.

As above U will only attain values in a sector of $\mathcal{T}$. This sector is described by the symbols

$$\{\mathbf{1}, X, \uparrow, \mathcal{I}(\curlyvee), \mathcal{I}(\Psi)\} \ .$$

Fortunately, it is *not* necessary to construct all of the processes in the list (4.2) by hand. Only the terms in the first set on the right hand side of (4.2) have to be constructed individually - the behaviour of the remaining ones follows by general principles.

In this case the action of the group Γ is *not* trivial.

4.2. **A VERY rough sketch of operations on regularity structures.** We now want to briefly describe the operations necessary to define (and solve) the enhanced fixed point problem for controlled distribution. As above we perform this analysis on the level of the Φ^4 equation. We want to define the Picard iteration

$$U_{n+1} = \Phi(U_n) = -\mathcal{K}(U_n^3 + \Xi) \ , \quad (4.3)$$

for $U \in \mathcal{D}^{\gamma}_{\alpha}$ taking values in a certain sector. Here $\mathcal{K}$ is supposed to represent the integration against the heat kernel on the level of the controlled distribution. Hence, we need to define the cube of a controlled

distribution as well as the operator $\mathcal{K}$. Both operations will be defined on the sectors where they are actually needed.

It turns out that as soon as one makes the right assumptions on the regularity structure and the model (which of course have to be verified) the operations turn into a purely formal exercise. Let us consider multiplication in the two dimensional Φ^4 model for example. If at some point x we have $U(x) = U_{\mathbf{1}}(x)\mathbf{1} + U_{\upharpoonright}(x)\upharpoonright$ then we simply define

$$U^3(x) = U_{\mathbf{1}}(x)^3\mathbf{1} + 3U_{\mathbf{1}}(x)^2U_{\upharpoonright}(x)\upharpoonright + 3U_{\mathbf{1}}(x)U_{\upharpoonright}(x)^2\Vee + U_{\upharpoonright}(x)^3\Vee\!\!\Vee\,.$$

The operation for the three dimensional case and for $U(x) = U_{\mathbf{1}}(x)\mathbf{1} + U_{\upharpoonright}(x)\upharpoonright + U_{\mathcal{I}\Vee}(x)\mathcal{I}\Vee + U_{\mathcal{I}\Vee\!\!\Vee}(x)\mathcal{I}\Vee$ is defined similarly, making use all of the symbols in the right hand side of (4.2). Fortunately, we can truncate just above order 1 so that quite a few terms drop.

The non-trivial ingredient is an extra assumption that the model is γ-regular (see [18, Def. 4.6]) this assumption states roughly speaking that on the level of the models multiplication and the spacial translations given by the action of Γ commute. If this condition is satisfied, then the formal multiplication described above is indeed a multilinear operation on the required $\mathcal{D}^\gamma$ spaces.

The definition of the integration map $\mathcal{K}$ is more subtle. A naive guess would be to simply define, for example $\mathcal{K}(\Vee\!\!\Vee) = \mathcal{I}(\Vee\!\!\Vee)$ but this definition does not work. Instead the $\mathcal{K}(\Vee)$ has a polynomial contribution and in three dimsionsions and it takes values in

$$\mathbf{R}\mathbf{1} + \mathbf{R}\mathcal{I}(\Vee)\,.$$

Unfortunately, a detailed discussion of this would take too long and goes beyond the scope of these lectures (see [18, Sec. 5] for a detailed discussion of this). Let us only point out that this effect introduces an interaction between the polynomials and the "extra terms" which were introduced by hand. It is responsible for the non-trivial action of the semigroup $\mathcal{G}$.

Last but not least, we have to mention the downward arrow depicted $\mathcal{R}$ in Figure 1. A modelled distribution gives a "local description" of a distribution up to a certain order. In the examples given at the beginning of Section 3 it was always trivial to construct the *Reconstruction operator* $\mathcal{R}$ that recovers the original distribution (function in those cases), but in general this is not true. The fact that as soon as γ is strictly positive, the reconstruction operator $\mathcal{R}$ can always be found and is unique, is the "raison d'être" for the definitions in Section 3. It is stated in [18, Thm 3.10].

5. Lecture 5

In this last lecture we explain a result on the connection of the two dimensional Φ^4 model and an Ising like model. This result was recently obtained in [24].

Similar result for spatial dimension 1 were proved in [5, 11] and our main result, Theorem 5.1 was already conjectured in [13]. The renormalisation ideas explained in the previous sections finally allowed to prove this conjecture. Although, as already explained above, the renormalisation of the Φ^4_2 equation does not strictly speaking require the notion of regularity structure, the proof is heavily inspired by techniques developed in [18].

In this section, following [24] we explain the necessary scaling.

5.1. **The microscopic model.** We are going to be interested in an Ising-like model with ferromagnetic interaction. The difference between our model and the "usual" Ising model is the fact that every particle interacts with all particles at distance $\lesssim \gamma^{-1}$. In this sense our model is an interpolation between the usual nearest neighbour Ising model and a mean field model.

Let us be more precise. For $N \geq 1$ let $\Lambda_N = \mathbf{Z}^2/(2N+1)\mathbf{Z}^2$ be the two-dimensional discrete torus which we identify with the set $\{-N, -(N-1), \dots, N\}^2$. Denote by $\Sigma_N = \{-1, +1\}^{\Lambda_N}$ the set of spin configurations on Λ_N. We will always denote spin configurations by $\sigma = (\sigma(k),\, k \in \Lambda_N)$.

Let $\mathfrak{K}\colon \mathbf{R}^2 \to [0,1]$ be a $\mathcal{C}^2$ function with compact support contained in $B(0,3)$, the Euclidean ball of radius 3 around 0 in $\mathbf{R}^2$. We assume that $\mathfrak{K}$ is invariant under rotations and satisfies

$$\int_{\mathbf{R}^2} \mathfrak{K}(x)\, dx = 1 \qquad \int_{\mathbf{R}^2} \mathfrak{K}(x)\, |x|^2\, dx = 4\,.$$

Then for $0 < \gamma < 1$ let $\kappa_\gamma\colon \Lambda_N \to [0,\infty)$ be defined as $\kappa_\gamma(0) = 0$ and

$$\kappa_\gamma(k) = c_{\gamma,1}\, \gamma^2\, \mathfrak{K}(\gamma k) \qquad k \neq 0\,,$$

where $c^{-1}_{\gamma,1} = \sum_{k\in\Lambda_N} \gamma^2\, \mathfrak{K}(\gamma k)$.

For any $\sigma \in \Sigma_N$ we introduce the locally averaged field

$$h_\gamma(\sigma, k) := \sum_{j\in\Lambda_N} \kappa_\gamma(k-j)\sigma(j) =: \kappa_\gamma * \sigma(k)\,, \tag{5.1}$$

and the Hamiltonian

$$\mathscr{H}_\gamma(\sigma) := -\frac{1}{2} \sum_{k,j\in\Lambda_N} \kappa_\gamma(k-j)\sigma(j)\sigma(k) = -\frac{1}{2} \sum_{k\in\Lambda_N} \sigma(k)\, h_\gamma(\sigma,k)\,. \tag{5.2}$$

In both (5.1) and (5.2) subtraction on Λ_N is to be understood with periodic boundary conditions. For any inverse temperature $\beta > 0$ we define the Gibbs measure λ_γ on Σ_N as

$$\lambda_\gamma(\sigma) := \frac{1}{\mathscr{Z}_\gamma} \exp\Big(-\beta \mathscr{H}_\gamma(\sigma)\Big)\,,$$

where as usual

$$\mathscr{Z}_\gamma := \sum_{\sigma\in\Sigma_N} \exp\Big(-\beta \mathscr{H}_\gamma(\sigma)\Big)\,,$$

denotes the normalisation constant that makes λ_γ a probability measure.

5.2. **Dynamics.** On Σ_N we study the Markov process given by the generator

$$\mathscr{L}_\gamma f(\sigma) = \sum_{j\in\Lambda_N} c_\gamma(\sigma,j)\big(f(\sigma^j)-f(\sigma)\big)\,, \tag{5.3}$$

acting on functions $f\colon \Sigma_N \to \mathbf{R}$. Here $\sigma^j \in \Sigma_N$ is the spin configuration that coincides with σ except for a flipped spin at position j. As jump rates $c_\gamma(\sigma,j)$ we choose those of the Glauber dynamics,

$$c_\gamma(\sigma,j) := \frac{\lambda_\gamma(\sigma^j)}{\lambda_\gamma(\sigma)+\lambda_\gamma(\sigma^j)}\,.$$

It is clear that these jump rates are reversible with respect to the measure λ_γ. Since $\kappa_\gamma(0)=0$, the local mean field $h_\gamma(\sigma,j)$ does not depend on $\sigma(j)$. Using also the fact that $\sigma(j)\in\{-1,1\}$, we can conveniently rewrite the jump rates as

$$\begin{aligned} c_\gamma(\sigma,j) &= \frac{e^{-\sigma(j)\beta h_\gamma(\sigma,j)}}{e^{\beta h_\gamma(\sigma,j)}+e^{-\beta h_\gamma(\sigma,j)}} \\ &= \frac{1}{2}\big(1-\sigma(j)\tanh\big(\beta h_\gamma(\sigma,j)\big)\big)\,. \end{aligned}$$

We write $(\sigma(t))_{t\ge0}$ for the (pure jump) Markov process on Σ_N thus defined, with the notation $\sigma(t)=(\sigma(t,k))_{k\in\Lambda_N}$. With a slight abuse of notation, we let

$$h_\gamma(t,k) = h_\gamma(\sigma(t),k)\,. \tag{5.4}$$

Our aim is to describe the critical behaviour of the local mean field h_γ as defined in (5.4), and to derive a non-linear SPDE for a suitably rescaled version of it. To this end we write, for $t\ge0$ and $k\in\Lambda_N$,

$$h_\gamma(t,k) = h_\gamma(0,k) + \int_0^t \mathscr{L}_\gamma\, h_\gamma(s,k)\,ds + m_\gamma(t,k)\,, \tag{5.5}$$

where the process $m_\gamma(\cdot,k)$ is a martingale. Observing that for any $\sigma\in\Sigma_N$ and for any $j,k\in\Lambda_N$, we have $h_\gamma(\sigma^j,k)-h_\gamma(\sigma,k) = -2\kappa_\gamma(k-j)\sigma(j)$, we get from (5.3) and (5.4)

$$\begin{aligned} \mathscr{L}_\gamma h_\gamma(\sigma,k) &= -h_\gamma(\sigma,k) + \kappa_\gamma * \tanh\big(\beta h_\gamma(\sigma,k)\big) \\ &= \Big(\kappa_\gamma * h_\gamma(\sigma,k) - h_\gamma(\sigma,k)\Big) + (\beta-1)\,\kappa_\gamma * h_\gamma(\sigma,k) \\ &\quad - \frac{\beta^3}{3}\Big(\kappa_\gamma * h_\gamma^3(\sigma,k)\Big) + \dots\,, \end{aligned} \tag{5.6}$$

where we have used the Taylor expansion $\tanh(\beta h) = \beta h - \frac{1}{3}(\beta h)^3 + \dots$.

The predictable quadratic covariations of the martingales $m_\gamma(\cdot,k)$ are given by

$$\langle m_\gamma(\cdot,k), m_\gamma(\cdot,j)\rangle_t = 4\int_0^t \sum_{\ell\in\Lambda_N} \kappa_\gamma(k-\ell)\,\kappa_\gamma(j-\ell)\,c_\gamma\big(\sigma(s),\ell\big)\,ds\,. \tag{5.7}$$

Furthermore, the jumps of $m_\gamma(\cdot,k)$ coincide with those of $h_\gamma(\cdot,k)$. In particular, if for some $\ell\in\Lambda_N$ the spin $\sigma(\ell)$ changes sign, then $m_\gamma(\cdot,k)$ has a jump of $-2\sigma(\ell)\kappa_\gamma(k-\ell)$.

5.3. **Rescaled dynamics.** For any $0 < \gamma < 1$ let $N = N(\gamma)$ be the microscopic system size determined below (in (5.12)). Then set $\varepsilon = \frac{2}{2N+1}$. Every *microscopic* point $k \in \Lambda_N$ can be identified with $x = \varepsilon k \in \Lambda_\varepsilon = \{x = (x_1, x_2) \in \varepsilon \mathbf{Z}^2 \colon -1 < x_1, x_2 < 1\}$. We view Λ_ε as a discretisation of the continuous two dimensional torus $\mathbb{T}^2$ identified with $[-1,1]^2$. For suitable scaling factors $0 < \alpha, \delta$ and inverse temperature β to be determined below we set

$$X_\gamma(t,x) = \frac{1}{\delta} h_\gamma\left(\frac{t}{\alpha}, \frac{x}{\varepsilon}\right) \qquad x \in \Lambda_\varepsilon,\, t \geq 0. \tag{5.8}$$

In these *macroscopic coordinates* the evolution equation (5.5) (together with (5.6)) reads

$$\begin{aligned} X_\gamma(t,x) =& X_\gamma(0,x) + \int_0^t \Big(\frac{\varepsilon^2}{\gamma^2}\frac{1}{\alpha} \Delta_\gamma X_\gamma(s,x) + \frac{(\beta-1)}{\alpha} K_\gamma *_\varepsilon X_\gamma(s,x) \\ & - \frac{\beta^3}{3}\frac{\delta^2}{\alpha} K_\gamma *_\varepsilon X_\gamma^3(s,x) + K_\gamma *_\varepsilon E_\gamma(s,x) \Big) ds + M_\gamma(t,x)\,, \end{aligned} \tag{5.9}$$

for $x \in \Lambda_\varepsilon$. Here we have set $K_\gamma(x) = \varepsilon^{-2}\kappa_\gamma(\varepsilon^{-1}x) = c_{\gamma,1}\frac{\gamma^2}{\varepsilon^2}\mathfrak{K}\big(\frac{\gamma}{\varepsilon}x\big)$ (the second equality being valid for $x \neq 0$). The convolution $*_\varepsilon$ on Λ_ε is defined through $X *_\varepsilon Y(x) = \sum_{z \in \Lambda_\varepsilon} \varepsilon^2 X(x-z) Y(z)$ (where subtraction on Λ_ε is to be understood with periodic boundary conditions on $[-1,1]^2$) and $\Delta_\gamma X = \frac{\gamma^2}{\varepsilon^2}(K_\gamma *_\varepsilon X - X)$ (so that Δ_γ scales like the continuous Laplacian). The rescaled martingale is defined as $M_\gamma(t,x) := \frac{1}{\delta} m_\gamma\Big(\frac{t}{\alpha}, \frac{x}{\varepsilon}\Big)$. Finally, the error term $E_\gamma(t,x)$ (implicit in (5.6)) is given by

$$E_\gamma(t,\cdot) = \frac{1}{\delta\alpha}\Big(\tanh\big(\beta\delta X_\gamma(t,\cdot)\big) - \beta\delta X_\gamma(t,\cdot) + \frac{(\beta\delta)^3}{3} X_\gamma(t,\cdot)^3 \Big)\,. \tag{5.10}$$

In these coordinates the expression (5.7) for the quadratic variation becomes

$$\begin{aligned} &\langle M_\gamma(\cdot,x), M_\gamma(\cdot,y)\rangle_t \\ &\qquad = 4\frac{\varepsilon^2}{\delta^2\alpha} \int_0^t \sum_{z \in \Lambda_\varepsilon} \varepsilon^2 K_\gamma(x-z)\, K_\gamma(y-z)\, C_\gamma\big(s,z\big)\, ds\,, \end{aligned} \tag{5.11}$$

where $C_\gamma(s,z) := c_\gamma(\sigma(s/\alpha), z/\varepsilon)$. In these macroscopic coordinates a spin flip at the microscopic position $k = \varepsilon^{-1}y$ causes a jump of $-2\sigma(\varepsilon^{-1}y)\delta^{-1}\varepsilon^2 K_\gamma(y-x)$ for the martingale $M_\gamma(\cdot,x)$.

The scaling of the approximated Laplacian, the term $K_\gamma *_\varepsilon X_\gamma^3$ and the quadratic variation in (5.11) suggest that in order to see non-trivial behaviour for these terms we need to impose

$$1 \approx \frac{\varepsilon^2}{\gamma^2}\frac{1}{\alpha} \approx \frac{\delta^2}{\alpha} \approx \frac{\varepsilon^2}{\delta^2\alpha}\,. \tag{5.12}$$

At first sight (5.9) suggests that β should be so close to one that $(\beta-1)/\alpha = O(1)$. But for $d \geq 2$ this naive guess is incorrect. As for the macroscopic equation the microscopic model has to be be renormalised. We set

$$(\beta - 1) = \alpha(\mathfrak{c}_\gamma - m)\,, \tag{5.13}$$

where the "mass" $m \in \mathbf{R}$ is fixed. The extra term $\mathfrak{c}_\gamma$ diverges logarithmically as γ goes to 0.

5.4. **Main result.** As above we denote by φ the solution of the dynamical Φ^4_2 equation

$$\partial_t \varphi = \Delta\varphi - \varphi^3 - m\varphi + \xi$$

(after renormalisation) for a fixed initial datum $\varphi_0 \in \mathcal{C}^{-\alpha}$. This process X is continuous taking values in $\mathcal{C}^{-\alpha}$.

Assume that for $\gamma > 0$ the spin configuration at time 0 is given by $\sigma_\gamma(0,k)$, $k \in \Lambda_N$ and define for $x \in \Lambda_\varepsilon$

$$X^0_\gamma(x) = \sum_{y \in \Lambda_\varepsilon} \varepsilon^2 K_\gamma(x-y)\sigma_\gamma(0, \varepsilon^{-1}y)\,.$$

Theorem 5.1. *Assume that the scaling assumption (5.12) as well as (5.13) hold. We extend X^0_γ and X_γ to arbitrary values $x \in \mathbb{T}^2$ in a suitable way. Then if X^0_γ converges to X_0 in $\mathcal{C}^{-\alpha}$ for $\alpha > 0$ small enough and that X_0, X^0_γ are uniformly bounded in $\mathcal{C}^{-\alpha+\bar\kappa}$ for an arbitrarily small $\bar\kappa > 0$. Then X_γ converges to φ in distribution with respect to the topology of $\mathcal{D}(\mathbf{R}_+, \mathcal{C}^{-\alpha})$.*

Remark 5.2. In principle one can perform the analysis that leads to (5.12) in any spatial dimension d. Indeed, the only necessary change is to replace the ε^2 appearing in (5.11) by ε^d. In this way one obtains the scaling relation

$$\varepsilon \approx \gamma^{\frac{4}{4-d}}\,, \qquad \alpha \approx \gamma^{\frac{2d}{4-d}}\,, \qquad \delta \approx \gamma^{\frac{d}{4-d}}\,. \tag{5.14}$$

This relation was already obtained in [13] and for $d = 1$ it is indeed the scaling used by [5, 11]. For $d = 3$ we expect that it is possible to combine the techniques developed in this article with the theory developed in [18] to get a convergence result to the dynamic Φ^4_3 model. For $d = 4$ relation (5.14) cannot be satisfied. This corresponds exactly to the fact that the Φ^4_4 model fails to satisfy the *subcriticality condition*.

References

[1] M. Aizenman. Geometric analysis of φ^4 fields and Ising models. I, II. *Comm. Math. Phys.*, 86(1):1–48, 1982.

[2] G. Amir, I. Corwin, and J. Quastel. Probability distribution of the free energy of the continuum directed random polymer in 1 + 1 dimensions. *Comm. Pure Appl. Math.*, 64(4):466–537, 2011.

[3] H. Bahouri, J.-Y. Chemin, and R. Danchin. *Fourier analysis and nonlinear partial differential equations*, volume 343. Springer, 2011.

[4] L. Bertini and G. Giacomin. Stochastic Burgers and KPZ equations from particle systems. *Comm. Math. Phys.*, 183(3):571–607, 1997.
[5] L. Bertini, E. Presutti, B. Rüdiger, and E. Saada. Dynamical fluctuations at the critical point: convergence to a nonlinear stochastic PDE. *Teor. Veroyatnost. i Primenen.*, 38(4):689–741, 1993.
[6] D. C. Brydges, J. Fröhlich, and A. D. Sokal. A new proof of the existence and nontriviality of the continuum φ 2 4 and φ 3 4 quantum field theories. *Communications in mathematical physics*, 91(2):141–186, 1983.
[7] I. Corwin. The kardar–parisi–zhang equation and universality class. *Random matrices: Theory and applications*, 1(01), 2012.
[8] I. Corwin and J. Quastel. Renormalization fixed point of the kpz universality class. *arXiv preprint arXiv:1103.3422*, 2011.
[9] G. Da Prato and A. Debussche. Strong solutions to the stochastic quantization equations. *Ann. Probab.*, 31(4):1900–1916, 2003.
[10] G. Da Prato and J. Zabczyk. *Stochastic equations in infinite dimensions*, volume 44 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1992.
[11] J. Fritz and B. Rüdiger. Time dependent critical fluctuations of a one-dimensional local mean field model. *Probab. Theory Related Fields*, 103(3):381–407, 1995.
[12] T. Funaki. Random motion of strings and related stochastic evolution equations. *Nagoya Mathematical Journal*, 89:129–193, 1983.
[13] G. Giacomin, J. L. Lebowitz, and E. Presutti. Deterministic and stochastic hydrodynamic equations arising from simple microscopic model systems. In *Stochastic partial differential equations: six perspectives*, volume 64 of *Math. Surveys Monogr.*, pages 107–152. Amer. Math. Soc., Providence, RI, 1999.
[14] J. Glimm and A. Jaffe. *Quantum physics*. Springer, 1981.
[15] J. Glimm, A. Jaffe, and T. Spencer. Phase transitions for φ 2 4 quantum fields. *Communications in Mathematical Physics*, 45(3):203–216, 1975.
[16] M. Gubinelli. Controlling rough paths. *J. Funct. Anal.*, 216(1):86–140, 2004.
[17] M. Hairer. An introduction to stochastic PDEs. *arXiv:0907.4178*, 2009.
[18] M. Hairer. A theory of regularity structures. *ArXiv e-prints*, Mar. 2013.
[19] M. Hairer. Solving the KPZ equation. *Annals of Mathematics*, 178(2):559–664, 2013.
[20] M. Hairer, M. D. Ryser, and H. Weber. Triviality of the 2d stochastic Allen-Cahn equation. *Electron. J. Probab*, 17(39):1–14, 2012.
[21] M. Kardar, G. Parisi, and Y.-C. Zhang. Dynamic scaling of growing interfaces. *Phys. Rev. Lett.*, 56(9):889–892, Mar 1986.
[22] N. Krylov. Lectures on elliptic and parabolic problems in hölder spaces. *Graduate Studies in Mathematics*, 12, 1998.
[23] T. J. Lyons. Differential equations driven by rough signals. *Rev. Mat. Iberoamericana*, 14(2):215–310, 1998.
[24] J.-C. Mourrat and H. Weber. Convergence of the two-dimensional dynamic Ising-Kac model to φ^4_2. in preparation.
[25] D. Nualart. *The Malliavin calculus and related topics*. Springer, 2006.
[26] R. Peierls. On ising's model of ferromagnetism. In *Mathematical Proceedings of the Cambridge Philosophical Society*, volume 32, pages 477–481. Cambridge Univ Press, 1936.
[27] C. Prévôt and M. Röckner. *A concise course on stochastic partial differential equations*, volume 1905 of *Lecture Notes in Mathematics*. Springer, Berlin, 2007.
[28] T. Sasamoto and H. Spohn. One-dimensional Kardar-Parisi-Zhang equation: an exact solution and its universality. *Phys. Rev. Lett.*, 104, 2010.

[29] K. A. Takeuchi and M. Sano. Universal fluctuations of growing interfaces: Evidence in turbulent liquid crystals. *Phys. Rev. Lett.*, 104:230601, Jun 2010.

Hendrik Weber, University of Warwick
E-mail address: hendrik.weber@warwick.ac.uk