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Infinite-dimensional stochastic differential equations and tail σ-fields

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Abstract: [][][]

1 Introduction

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We study infinite-dimensional stochastic differential equations (ISDEs) of $\mathbf{X} \in C([0,\infty);(\mathbb{R}^d)^{\mathbb{N}})$ describing infinitely many Brownian particles moving in $\mathbb{R}^d$ with free potential $\Phi = \Phi(x)$ and interacting potential $\Psi = \Psi(x,y)$. The ISDEs of $\mathbf{X} = (X^i)_{i\in\mathbb{N}}$ are then given by

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$$dX_t^i = dB_t^i - \frac{\beta}{2}\nabla_x \Phi(X_t^i)dt - \frac{\beta}{2}\sum_{j\neq i}^{\infty} \nabla_x \Psi(X_t^i, X_t^j)dt. \tag{1.1}$$

Here $\nabla_x = (\frac{\partial}{\partial x_i})_{i=1}^d$ is the nabla in x and β is the inverse temparature.

The study of ISDEs are initiated by Lang [21, 22] (lang.1,lang.2), followed by Fritz [8] (Fr), and Tanemura [46] (T2), and Tanemura-Roelly []. In these works, the free potentials Φ are assumed to be zero and interaction potentials Ψ are of class $C_0^3(\mathbb{R}^d)$ or exponentially decay at infinity. Such a restriction on Ψ exclude polynomial decay and logarithmic growth interaction potentials, which are extremely significant from view point of statistical physics and random matrix theory. Examples such ISDEs are the following:

Sine$_\beta$ interacting Brownian motions (Dyson model infinite dimensions) with $\beta = 1, 2, 4$: We set $d = 1$, $\Phi(x) = 0$, $\Psi(x,y) = -\log|x-y|$, and intoroduce the ISDE

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$$dX_t^i = dB_t^i + \frac{\beta}{2}\lim_{r\to\infty}\sum_{|X_t^i - X_t^j|<r,\ j\neq i} \frac{1}{X_t^i - X_t^j}dt \quad (i \in \mathbb{N}). \tag{1.2}$$

Airy$_\beta$ interacting Brownian motions ($\beta = 1, 2, 4$): We set $d = 1$, $\Phi(x) = 0$, and introduce the ISDE

$$dX_t^i = dB_t^i + \frac{\beta}{2} \lim_{r\to\infty} \Big\{ \Big(\sum_{j\neq i,\ |X_t^j|<r} \frac{1}{X_t^i - X_t^j} \Big) - \int_{|x|<r} \frac{\hat{\rho}(x)}{-x} dx \Big\} dt. \tag{1.3}$$

Here we set

$$\hat{\rho}(x) = \frac{1_{(-\infty,0)}(x)}{\pi} \sqrt{-x}. \tag{1.4}$$

Ginibre interacting Brownian motions ($\beta = 2$): We set $d = 2$, $\Psi(x, y) = -\log|x-y|$, and introduce the ISDE

$$dX_t^i = dB_t^i + \lim_{r\to\infty} \sum_{|X_t^i - X_t^j|<r,\ j\neq i} \frac{X_t^i - X_t^j}{|X_t^i - X_t^j|^2} dt \quad (i \in \mathbb{N}) \tag{1.5}$$

and also

$$dX_t^i = dB_t^i - X_t^i + \lim_{r\to\infty} \sum_{|X_t^j|<r,\ j\neq i} \frac{X_t^i - X_t^j}{|X_t^i - X_t^j|^2} dt \quad (i \in \mathbb{N}). \tag{1.6}$$

All these examples are arising from random matrix theory. The ISDEs (1.2) and (1.3) are the dynamical bulk and soft edge scaling limits of the finite particle systems of Gaussian orthogonal/unitary/symplectic ensembles, respectively. The ISDEs (1.5) and (1.6) are dynamical bulk scaling limits of Ginibre ensemble, which is a system of eigen values of non-Hermitian Gaussian random matrices. We will prove the ISDEs (1.5) and (1.6) have the same strong solutions, which reflects the dynamical rigidity of two dimensional stochastic Coulomb systems.

A classical example is the Lennard-Jones 6-12 potential. Let $d = 3$, $\beta > 0$, and $\Psi_{6,12}(x) = \{|x|^{-12} - |x|^{-6}\}$. The interaction $\Psi_{6,12}$ is called the Lennard-Jones 6-12 potential. The corresponding ISDE is:

$$dX_t^i = dB_t^i + \frac{\beta}{2} \sum_{j=1, j\neq i}^{\infty} \{ \frac{12(X_t^i - X_t^j)}{|X_t^i - X_t^j|^{14}} - \frac{6(X_t^i - X_t^j)}{|X_t^i - X_t^j|^{8}} \} dt \quad (i \in \mathbb{N}). \tag{1.7}$$

Although the potential $\Psi_{6,12}$ is of Ruelle's class, this example is also excluded by the previous result because of the polynomial decay at infinity. Since the sum in (1.7) absolutely convergent, we do not put the prefactor $\lim_{r\to\infty}$ unlike other examples (1.2)–(1.6). We refer to Section 2 for further examples.

In previous works, the Itô scheme was used to solve the ISDEs, which requires a good control of (at least local) Lipschitz continuity of coefficients. Because the state space $(\mathbb{R}^d)^{\mathbb{N}}$ is huge, such a localization is quite hard to carry out. As a result, known results are far from the ideal. Indeed, no examples above can not be covered by the previous results.

In the present paper, we introduce a fresh method to establish the existence and the pathwise uniqueness of strong solutions of the ISDEs including these ISDEs with long range potentials. Our results are applicable to surprisingly wide range of the models and, in particular, all the examples above and Section 1.

When one prove the unique, existence of strong solutions of ISDEs, the usage of Itô scheme is indispensable, which causes the difficulty to treat the ISDEs in infinite dimensions. We will not use the Itô scheme ISDEs *directly*, but use it infinitely many times to an infinite system of finite-dimensional ISDEs with consistency (IFC), which we explain in the sequel.

Our method is based on several novel ideas, and divided into mainly two steps. The first step begins by reducing the ISDE to a differential equation of random point field μ satisfying

$$\nabla_x \log \mu^{[1]}(x,\mathsf{s}) = -\frac{\beta}{2}\big\{\nabla_x \Phi(x) + \lim_{r\to\infty} \sum_{|s_j|<r}^{\infty} \nabla_x \Psi(x-s_j)\big\}. \tag{1.8}$$

Here $\mathsf{s}=\sum_i \delta_{s_i}$, $\mu^{[1]}$ is the one-Campbell measure of μ defined by (2.9), and $\nabla_x \log \mu^{[1]}$ defined by (2.11). We call $\nabla_x \log \mu^{[1]}$ the logarithmic derivative of μ. The equation (1.8) is given in an informal manner here, and we refer to Section 2 for the precise definition.

The first author proved in [31] that, if (1.8) has a solution μ satisfying the assumptions **(A2)**–**(A5)** in Section 2, then the ISDE (1.1) has a weak solution $(\mathbf{X},\mathbf{B})$ starting at $\mathbf{s}=(s_i)_{i\in\mathbb{N}}$. Here a weak solution means the pair of the stochastic process $\mathbf{X}$ and the $(\mathbb{R}^d)^{\mathbb{N}}$-valued Brownian motion $\mathbf{B}$ satisfying (1.1).

By using the weak solution $(\mathbf{X},\mathbf{B})$, we introduce the infinite system of finite-dimensional stochastic differential equations with consistency (IFC), namely, a family of SDEs of $\mathbf{Y}^m=(Y^{m,i})_{i=1}^n$ given by

$$\begin{aligned} dY_t^{m,i} &= dB_t^i - \frac{\beta}{2}\nabla_x\Phi(Y_t^{m,i})dt - \frac{\beta}{2}\sum_{j\neq i}^{m}\nabla_x\Psi(Y_t^{m,i}-Y_t^{m,j})dt \\ &\qquad - \frac{\beta}{2}\lim_{r\to\infty}\sum_{\substack{j=m+1,\\ |X_t^{m,j}|<r}}^{\infty}\nabla_x\Psi(Y_t^{m,i}-X_t^{m,j})dt, \\ \mathbf{Y}_0^m &= \mathbf{s}_m. \end{aligned} \tag{1.9}$$

Here for each $m\in\mathbb{N}$ we set $\mathbf{X}^{m*}=(X^i)_{i=m+1}^{\infty}$, $\mathbf{s}_m=(s_1,\ldots,s_m)$ for $\mathbf{s}=(s_i)_{i=1}^{\infty}$, and $\mathbf{B}^m=(B^1,\ldots,B^m)$ is the $(\mathbb{R}^d)^m$-valued Brownian motions. We regard $\mathbf{X}^{m*}=(X^i)_{i=m+1}^{\infty}$ as a part of coefficients of the SDE (1.9), and the SDEs (1.9) are time inhomogeneous. Under suitable assumptions, the SDE then has a strong solution $\mathbf{Y}^{\mathbf{m}}$ and the pathwise uniqueness. Hence $\mathbf{Y}^m$ is a function of $\mathbf{s}_m$, $\mathbf{B}^m$, and $\mathbf{X}$. As a function of $\mathbf{X}$, the solution $\mathbf{Y}^m$ depends only on $\mathbf{X}^{m*}$. Hence $\mathbf{Y}^m$ is $\mathcal{B}((\mathbb{R}^d)^m)\times\sigma[\mathbf{B}^m]\times\sigma[\mathbf{X}^{m*}]$-measurable. We thus write

$$\mathbf{Y}^m = \mathbf{Y}^m(\mathbf{s}_m,\mathbf{B}^m,\mathbf{X}^{m*}).$$

By the pathwise uniqueness we see that $\mathbf{X}^m = (X^1, \dots, X^m)$ is the unique strong solution of (1.9). Hence we deduce that

$$\mathbf{X}^m = \mathbf{Y}^m(\mathbf{s}_m, \mathbf{B}^m, \mathbf{X}^{m*}) \tag{1.10}$$

This implies that $\mathbf{X}^m$ becomes a function of $\mathbf{s}_m$, $\mathbf{B}^m = (B^1, \dots, B^d)$, and, furthermore, $\mathbf{X}^{m*} = (X^{m+1}, , \dots)$. The last dependence is through the coefficients of the SDE (1.9).

The relation (1.10) is the crucial consistency we use. From this we deduce that $\mathbf{Y}^m$ has a limit $\mathbf{X}$ as m goes to infinity, and that $\mathbf{X}$ is a functional of $\mathbf{s}$, $\mathbf{B}$, and $\mathbf{X}$ itself through $\{\mathbf{X}^{m*}\}_{m\in\mathbb{N}}$. The point here is that as a functional of $\mathbf{X} = (X^1, X^2, \dots)$, the limit $\mathbf{X}$ is measurable with respect to the tail σ-field

$$\mathrm{Tail}_{\mathrm{path}} = \bigcap_{m=1}^{\infty} \sigma[\mathbf{X}^{m*}]. \tag{1.11}$$

Hence the existence of the strong solutions and their pathwise uniqueness is reduced to the triviality of the labeled tail σ-field $\mathrm{Tail}_{\mathrm{path}}$ with respect to the regular conditional distribution $\mathbf{P}_{\mathbf{s}}(\,\cdot\,|\mathbf{B}, \mathbf{s})$ of the distribution $\mathbf{P}_{\mathbf{s}}$ of the weak solution $(\mathbf{X}, \mathbf{B})$ starting at $\mathbf{s}$.

In Theorem 3.4, we will give a necessary and sufficient condition of the existence of the strong solutions and the pathwise uniqueness in terms of the property of $\mathrm{Tail}_{\mathrm{path}}$ (First tail theorem). The key point here is, to some extent, we regard as the labeled tail σ-field $\mathrm{Tail}_{\mathrm{path}}$ as a *boundary condition* of the ISDE (1.1).

In the second step, we prove the triviality of $\mathrm{Tail}_{\mathrm{path}}$ with respect to the distribution $\mathbf{P}_{\mathbf{s}}(\,\cdot\,|\mathbf{B}, \mathbf{s})$. Since $C([0,\infty); (\mathbb{R}^d)^{\mathbb{N}})$ is a huge space and the tail σ-field is not topologically well behaved, this step is quite hard. Our key point here is that, under absolutely continuity condition (2.35), the triviality of $\mathrm{Tail}_{\mathrm{path}}$ mentioned above follows from that of the tail σ-field of the configuration space over $\mathbb{R}^d$ with respect to the random point field μ (Second tail theorem). Furthermore, even if the random point field μ is not tail trivial, we can decompose it to the tail trivial random point fields and solve the ISDEs in this case as well.

The organization of the paper is as follows: In Section 2, we give a set up and state the main theorems (Theorems 2.1 and 2.2), and in addition, give various examples. In Section 3, we introduce the notion of IFC solutions of the ISDEs, and clarify the relation between the strong solutions and the IFC solutions in Theorem 3.4. We will do this in a quite general setting beyond the interacting Brownian motions. In Section 4 we derive the tail triviality of $C_T(S^{\mathbb{N}})$ from the cylindrical tail triviality of $C_T(S^{\mathbb{N}})$. In Section 5 we derive the cylindrical tail triviality of $C_T(S^{\mathbb{N}})$ from the tail triviality of the configuration space S with respect to the random point field μ. In Section 6 we prove the first main theorem Theorem 2.1. In Section 7 we prove the second main theorem Theorem 2.2. In Section 8 we give sufficient conditions of Assumptions **(A4)**–**(A8)**, which are the main assumptions in Theorems 2.1 and 2.2. In Section 9 we prove Theorem 2.3. We prove the existence and the pathwise uniqueness of the strong solutions of the ISDEs (1.2) of sine$_\beta$ interacting Brownian motions with $\beta = 1, 2, 4$. In Section 10 we prove Theorem 2.5. We prove that of the ISDEs (1.5) and (1.6) of Ginibre interacting Brownian motions. In Section 11 we prove Theorem 2.6. This theorem is a genera result valid for all Ruelle's class potentials with marginal assumptions (Example 2.4). As particular examples of this class, we solve ISDEs (2.61) and (2.62) with the Lennard-Jones 6-12 potential (Example 2.5) and Riesz potentials (Example 2.6).

2 Set up and main results

Let S be a closed in $\mathbb{R}^d$ whose interior S_{int} is an connected open set satisfying that $\overline{S_{\text{int}}} = S$ and that the boundary ∂S has Lebesgue measure zero. We will solve ISDEs on $S^{\mathbb{N}}$. To formulate the ISDEs we recall the notion of the configuration space over S.

A configuration $\mathsf{s} = \sum_i \delta_{s_i}$ on S is a Radon measure on S consisting of delta masses. Let S be the configuration space over S. Let $S_r = \{s \in S\, ;\, |s| \le r\}$. Then by definition S is the set given by

$$(2.1)\qquad \mathsf{S} = \{\mathsf{s} = \sum_i \delta_{s_i}\, ;\ \ \mathsf{s}(S_r) < \infty \text{ for all } r \in \mathbb{N}\}.$$

By convention we regard the zero measure as an element of S. It is known that S is a Polish space equipped with the vague topology,

We will study stochastic differential equations on $S^{\mathbb{N}}$. To give relations among S, $S^{\mathbb{N}}$, and related spaces, we introduce unlabel maps $\mathfrak{u}_m : S^m \times \mathsf{S} \to \mathsf{S}$ $(m \in \mathbb{N} \cup \{\infty\})$ such that

$$(2.2)\qquad \mathfrak{u}_m((\mathbf{x}, \mathsf{y})) = \sum_{i=1}^m \delta_{x_i} + \mathsf{y} \quad \text{for } \mathbf{x} = (x_i) \in S^m, \quad \mathsf{y} \in \mathsf{S}.$$

By the same symbol $\mathfrak{u}_m$, we also denote the map $\mathfrak{u}_m : S^m \to \mathsf{S}$ such that $\mathfrak{u}(\mathbf{x}) = \sum_i \delta_{x_i}$, where $\mathbf{x} = (x_i)$. We often omit m if no confusion occurs. A measurable map $\mathfrak{l} : \mathsf{S} \to S^{\mathbb{N}}$ is called a label if $\mathfrak{u} \circ \mathfrak{l}(\mathsf{s}) = \mathsf{s}$ for all $\mathsf{s} \in \mathsf{S}$. There exist many labels $\mathfrak{l}$. We will fix one throughout the paper.

Let $\mathbf{X} = \{\mathbf{X}_t\}_{t \in [0,\infty)}$ be a labeled process such that

$$(2.3)\qquad \mathbf{X} = (X^i)_{i \in \mathbb{N}} \in C([0, \infty); S^{\mathbb{N}}).$$

For $\mathbf{X}$ and $i \in \mathbb{N}$ we define the unlabeled processes $\mathsf{X} = \{\mathsf{X}_t\}$ and $\mathsf{X}^{i\diamondsuit} = \{\mathsf{X}_t^{i\diamondsuit}\}$ as

$$(2.4)\qquad \mathsf{X}_t = \sum_{i \in \mathbb{N}} \delta_{X_t^i}, \quad \mathsf{X}_t^{i\diamondsuit} = \sum_{j \in \mathbb{N},\ j \ne i} \delta_{X_t^j}.$$

Let $\widetilde{\mathsf{S}}$ be a measurable subset of S. Define the set $\widetilde{\mathsf{S}}^{[1]}$ by $\widetilde{\mathsf{S}}^{[1]} = \mathfrak{u}_1^{-1}(\widetilde{\mathsf{S}}) \subset S \times \mathsf{S}$. Let $\sigma : \widetilde{\mathsf{S}}^{[1]} \to \mathbb{R}^{d^2}$ and $b : \widetilde{\mathsf{S}}^{[1]} \to \mathbb{R}^d$ be measurable functions. We consider the following ISDE of $\mathbf{X} = (X^i)_{i \in \mathbb{N}}$.

$$dX_t^i = \sigma(X_t^i, \mathsf{X}_t^{i\diamondsuit})dB_t^i + b(X_t^i, \mathsf{X}_t^{i\diamondsuit})dt \quad (i \in \mathbb{N}), \tag{2.5}$$

$$\mathsf{X}_t \in \mathsf{H} \quad \text{for all } t \in [0, \infty), \tag{2.6}$$

$$\mathbf{X}_0 = \mathbf{s}. \tag{2.7}$$

Here $\mathbf{B} = (B^i)_{i \in \mathbb{N}}$ is the $(\mathbb{R}^d)^{\mathbb{N}}$-valued standard Brownian motion, namely, $\{B^i\}_{i \in \mathbb{N}}$ are independent copies of d-dimensional Brownian motions. H is a subset of S such that $\sigma(x, \mathsf{s})$ and $b(x, \mathsf{s})$ take finite value for all $(x, \mathsf{s}) \in \mathsf{H}^{[1]}$. By (2.6) the initial starting point $\mathbf{s}$ in (2.7) is supposed to satisfy $\mathsf{s} = \mathfrak{u}(\mathbf{s}) \in \mathsf{H}$, where $\mathsf{s} = \sum_{i=1}^{\infty} \delta_{s_i}$ and $\mathbf{s} = (s_i)_{i \in \mathbb{N}}$ by definition.

In infinite dimensions, it is natural to consider coefficients σ and b defined only a suitable subset. Hence to detect the sufficiently large subset H such that (2.5)–(2.7) for all $\mathfrak{u}(\mathbf{s}) \in \mathsf{H}$ is an important step to solve the ISDEs. We emphasize that (2.5)–(2.7) are equations of $\mathbf{X}$ and H.

A pair of stochastic processes $(\mathbf{X}, \mathbf{B})$ is called a weak solution of (2.5)–(2.7) if for suitable H the pair $(\mathbf{X}, \mathbf{B})$ satisfies the ISDE (2.5)–(2.7) for each $\mathbf{s} \in \mathfrak{l}(\mathsf{H})$. We call $\mathbf{X}$ is a strong solution of (2.5)–(2.7) if, in addition, $\mathbf{X}$ is a functional of given Brownian motion $\mathbf{B}$. Weak solutions are often called solutions. In the present paper, we call them weak solutions to distinguish them from strong solutions. We refer to [13] and [41] for the definition of strong solutions.

In the present paper we will deduce the existence of a unique, strong solution of the ISDE (2.5)–(2.7) from an existence of a random point field μ satisfying the assumptions below. To state the assumptions on μ, we recall the notion of the logarithmic derivative, the quasi-Gibbs property, and related notions.

A symmetric and locally integrable function $\rho^n : S^n \to [0, \infty)$ is called the n-point correlation function of a random point field μ on S with respect to the Lebesgue measure if ρ^n satisfies

$$\int_{A_1^{k_1} \times \cdots \times A_m^{k_m}} \rho^n(x_1, \ldots, x_n) dx_1 \cdots dx_n = \int_{\mathsf{S}} \prod_{i=1}^{m} \frac{\mathsf{s}(A_i)!}{(\mathsf{s}(A_i) - k_i)!} d\mu \tag{2.8}$$

for any sequence of disjoint bounded measurable sets $A_1, \ldots, A_m \in \mathcal{B}(S)$ and a sequence of natural numbers $k_1, \ldots, k_m$ satisfying $k_1 + \cdots + k_m = n$. When $\mathsf{s}(A_i) - k_i < 0$, according to our interpretation, $\mathsf{s}(A_i)!/(\mathsf{s}(A_i) - k_i)! = 0$ by convention.

A Radon measure $\mu^{[1]}$ on $S \times \mathsf{S}$ is called the 1-Campbell measure of μ if $\mu^{[1]}$ is given by

$$\mu^{[1]}(dx d\mathsf{s}) = \rho^1(x) \mu_x(d\mathsf{s}) dx. \tag{2.9}$$

Here ρ^1 is the one correlation function of μ with respect to the Lebesgue measure, and μ_x is the reduced Palm measure conditioned at x. Namely, μ_x is the regular conditional probability defined by

$$\mu_x = \mu(\cdot - \delta_x | \mathsf{s}(\{x\}) \ge 1). \tag{2.10}$$

For a subset A we set $\pi_A : \mathsf{S} \to \mathsf{S}$ by $\pi_A(\mathsf{s}) = \mathsf{s}(\cdot \cap A)$. We say a function f on S is local if f is $\sigma[\pi_K]$-measurable for some compact set K in S. We also say f is smooth if $\check{f}$ is smooth, where $\check{f}(x_1, \ldots)$ is the symmetric function such that $\check{f}(x_1, \ldots) = f(\mathsf{x})$ for $\mathsf{x} = \sum_i \delta_{x_i}$. Let $\mathcal{D}_\circ$ be the set of all bounded, local smooth functions on S.

A $\mathbb{R}^d$-valued function $\mathsf{d}^\mu \in L^1_{\mathrm{loc}}(\mu^{[1]})$ is called the logarithmic derivative of μ if for all $\varphi \in C_0^\infty(S) \otimes \mathcal{D}_\circ$

$$\int_{S \times \mathsf{S}} \mathsf{d}^\mu(x, \mathsf{y}) \varphi(x, \mathsf{y}) \mu^{[1]}(dx d\mathsf{y}) = - \int_{S \times \mathsf{S}} \nabla_x \varphi(x, \mathsf{y}) \mu^{[1]}(dx d\mathsf{y}). \tag{2.11}$$

If the boundary ∂S is nonempty and particles hit the boundary, then d^μ would contain the term arising from the boundary condition. For example, if Neumann boundary condition is posed on the boundary, then local time type drifts would exist. In this sense, it would be more natural to suppose that d^μ are distributions rather than $\mathsf{d}^\mu \in L^1_{\mathrm{loc}}(\mu^{[1]})$. Instead, we will later assume that particles never hit the boundary, so the formulation as above is sufficient in the present situation. It would be interesting to generalize the theory including the case with boundary condition. We do not however pursue this here.

A sufficient condition for the explicit expression of the logarithmic derivative of random point fields is given in [31, Theorem 45]. Using this, one can obtain the logarithmic derivative of random point fields appearing in random matrix theory such as sine_β, Airy_β ($\beta = 1, 2, 4$), and the Ginibre random point field (see Examples 2.1–2.4 below). In the case of canonical Gibbs measures with Ruelle's class interaction potentials, one can easily calculate the logarithmic derivative through Dobrushin-Lanford-Ruelle equation (see Lemma 11.1).

To relate the ISDE (2.5) with the random point field μ, we assume:

(A1) For σ and $b \in L^{[1]}_{\mathrm{loc}}(\mu^{[1]})$, there exists a random point field μ with a logarithmic derivative $\mathsf{d}^\mu = \mathsf{d}^\mu(x, \mathsf{y})$ satisfying the relation

$$b(x, \mathsf{y}) = \frac{1}{2} \{ \nabla_x a(x, \mathsf{y}) + a(x, \mathsf{y}) \mathsf{d}^\mu(x, \mathsf{y}) \}. \tag{2.12}$$

Here ∇_x is the nabla in $\mathbb{R}^d$ and $a = a(x, \mathsf{y})$ is the $d \times d$-matrix valued function defined by

$$a(x, \mathsf{y}) = \sigma(x, \mathsf{y})^t \sigma(x, \mathsf{y}). \tag{2.13}$$

Furthermore, we assume that there exists a smooth positive function $a_0 : \mathbb{R}^+ \to \mathbb{R}^+$, and positive constants c_1 and c_2 such that

$$c_1 a_0(|x|) |\xi|^2 \le (a(x, \mathsf{y}) \xi, \xi)_{\mathbb{R}^d} \le c_2 a_0(|x|) |\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^d. \tag{2.14}$$

Remark 2.1. Taking (2.11) into account, we obtain an informal expression $\mathsf{d}^{\mu} = \nabla \log \mu^{[1]}$. Then we interpret the relation (2.12) as a differential equation of the random point field μ as follows:

$$b(x,\mathsf{y}) = \frac{1}{2}\{\nabla_x a(x,\mathsf{y}) + a(x,\mathsf{y})\nabla_x \log \mu^{[1]}\}. \tag{2.15}$$

Namely, (2.15) is an equation of μ for given coefficients σ and b. We will prove in Theorem 2.1 and Theorem 2.2 that, if the differential equation (2.15) of random point fields has a solution μ satisfying the assumptions **(A2)**–**(A8)** below, then the ISDE (2.5)–(2.7) has a unique strong solution.

Let Λ_r be the Poisson random point field whose intensity is the Lebesgue measure on S_r and set $\Lambda_r^m = \Lambda_r(\cdot \cap \mathsf{S}_r^m)$, where $\mathsf{S}_r^m = \{\mathsf{s} \in \mathsf{S}\, ;\, \mathsf{s}(S_r) = m\}$.

A random point field μ is called a (Φ,Ψ)-quasi Gibbs measure if its regular conditional probabilities

$$\mu_{r,\xi}^m = \mu(\,\pi_{S_r} \in \cdot \,|\, \pi_{S_r^c}(\mathsf{x}) = \pi_{S_r^c}(\xi),\, \mathsf{x}(S_r) = m)$$

satisfy, for all $r, m \in \mathbb{N}$ and μ-a.s. ξ,

$$c_1^{-1} e^{-\mathcal{H}_r(\mathsf{x})} \Lambda_r^m(d\mathsf{x}) \le \mu_{r,\xi}^m(d\mathsf{x}) \le c_1 e^{-\mathcal{H}_r(\mathsf{x})} \Lambda_r^m(d\mathsf{x}). \tag{2.16}$$

Here $c_3 = c_3(r, m, \xi)$ is a positive constant depending on r, m, ξ. For two measures μ, ν on a σ-field $\mathcal{F}$ we write $\mu \le \nu$ if $\mu(A) \le \nu(A)$ for all $A \in \mathcal{F}$. Moreover, $\mathcal{H}_r$ is the Hamiltonian on S_r defined by

$$\mathcal{H}_r(\mathsf{x}) = \sum_{x_i \in S_r} \Phi(x_i) + \sum_{j \ne k,\, x_j, x_k \in S_r} \Psi(x_j, x_k) \quad \text{ for } \mathsf{x} = \sum_i \delta_{x_i}.$$

We remark that, if μ is a (Φ,Ψ)-quasi Gibbs measure, then μ is also $(\Phi + \Phi_{\mathrm{loc,bdd}}, \Psi)$-quasi Gibbs measure for any locally bounded measurable function $\Phi_{\mathrm{loc,bdd}}$. In this sense, the notion of "quasi-Gibbs" is robust for the perturbation of free potentials. In particular, as we see later, both of the sine_β and the Airy_β random point fields are $(0, -\beta \log|x-y|)$-quasi Gibbs measures, where $(\beta = 1, 2, 4)$.

We assume:

(A2) μ is a (Φ,Ψ)-quasi Gibbs measure such that there exist upper semi-continuous functions $(\hat{\Phi}, \hat{\Psi})$ and positive constants c_4 and c_5 satisfying

$$c_4^{-1}\hat{\Phi}(x) \le \Phi(x) \le c_4 \hat{\Phi}(x), \quad c_5^{-1}\hat{\Psi}(x) \le \Psi(x) \le c_5 \hat{\Psi}(x). \tag{2.17}$$

We refer to [32, 33] for sufficient conditions of **(A2)**. These conditions give us quasi-Gibbs property of random point fields appearing in random matrix theory, such as sine_β, Airy_β $(\beta = 1, 2, 4)$, Bessel_2, and Ginibre random point fields [32, 33, 35, 12]. In the case of canonical Gibbs measures, the quasi-Gibbs property is obvious from the Dobrushin-Lanford-Ruelle equation.

Let σ_r^k be the k-density function of μ on S_r with respect to the Lebesgue measure $d\mathbf{x}_k$ on S_r^k. By definition σ_r^k is the non-negative symmetric function such that

$$\frac{1}{k!}\int_{S_r^k} \check{f}(\mathbf{x}_k)\sigma_r^k(\mathbf{x}_k)d\mathbf{x}_k = \int_{\mathsf{S}_r^k} f(\mathsf{x})\mu(d\mathsf{x}) \quad \text{for all } f \in C_b(\mathsf{S}). \tag{2.18}$$

Here we set $\mathbf{x}_k = (x_1, \ldots, x_k)$ and $d\mathbf{x}_k = dx_1 \cdots dx_k$. We assume:

(A3) μ satisfies the following:

$$\sum_{k=1}^{\infty} k\mu(\mathsf{S}_r^k) < \infty \quad \text{for all } r \in \mathbb{N}, \tag{2.19}$$

$$\sigma_r^k \in L^p(S_r^k, d\mathbf{x}^k) \quad \text{for all } k, r \in \mathbb{N} \text{ for some } 1 < p \le \infty. \tag{2.20}$$

Let $(\mathcal{E}^{a,\mu}, \mathcal{D}_\circ^{a,\mu})$ be a bilinear form on $L^2(\mu)$ with domain $\mathcal{D}_\circ^{a,\mu}$ defined by

$$\mathcal{D}_\circ^{a,\mu} = \{f \in \mathcal{D}_\circ \cap L^2(\mathsf{S}, \mu)\,;\, \mathcal{E}^{a,\mu}(f,f) < \infty\}, \tag{2.21}$$

$$\mathcal{E}^{a,\mu}(f,g) = \int_{\mathsf{S}} \mathbb{D}^a[f,g]\,\mu(d\mathsf{s}), \tag{2.22}$$

$$\mathbb{D}^a[f,g](\mathsf{s}) = \frac{1}{2}\sum_i (a(s_i, \mathsf{s}^{i\diamondsuit})\nabla_{s_i}\check{f}, \nabla_{s_i}\check{g})_{\mathbb{R}^d}. \tag{2.23}$$

Here $\mathsf{s} = \sum_i \delta_{s_i}$ and $\mathsf{s}^{i\diamondsuit} = \sum_{j\neq i}\delta_{s_j}$, ∇_{s_i} denotes the nabla $\nabla_{s_i} = (\frac{\partial}{\partial s_{i1}}, , \ldots, \frac{\partial}{\partial s_{id}})$ and $(\cdot,\cdot)_{\mathbb{R}^d}$ denotes the standard inner product in $\mathbb{R}^d$. When a is the unit matrix, we often suppress it from the notation, that is, $\mathcal{E}^{a,\mu} = \mathcal{E}^{\mu}$, $\mathcal{D}_\circ^{a,\mu} = \mathcal{D}_\circ^{\mu}$, $\mathbb{D}^a = \mathbb{D}$, and so on.

A family of probability measures $\{\mathsf{P}_\mathsf{s}\}_{\mathsf{s}\in\mathsf{S}}$ on $C([0,\infty);\mathsf{S})$ is called a conservative diffusion if the canonical process $\mathsf{X} = \{\mathsf{X}_t\}$ under P_s is a continuous process $\mathsf{X} = \{\mathsf{X}_t\}$ with the strong Markov property starting at s.

From **(A2)** we deduce that the non-negative form $(\mathcal{E}^{a,\mu}, \mathcal{D}_\circ^{a,\mu})$ is closable on $L^2(\mathsf{S},\mu)$ ([28, 32]). So let $(\mathcal{E}^{a,\mu}, \mathcal{D}^{a,\mu})$ be its closure on $L^2(\mathsf{S},\mu)$. Then from **(A3)** and $1 \in \mathcal{D}^{a,\mu}$ we deduce that there exists a conservative diffusion $\{\mathsf{P}_\mathsf{s}\}_{\mathsf{s}\in\mathsf{S}}$ associated with the Dirichlet form $(\mathcal{E}^{a,\mu}, \mathcal{D}^{a,\mu})$ on $L^2(\mathsf{S},\mu)$ (see [28]). In [28, (A.2)], we assumed that $\sigma_r^k \in L^\infty(S_r^k, d\mathbf{x}^k)$ for all $k, r \in \mathbb{N}$. We can relax this assumption as (2.20) with the argument in the proof of Lemma 8.5.

Since $1 \in \mathcal{D}^{a,\mu}$, the diffusion $\{\mathsf{P}_\mathsf{s}\}_{\mathsf{s}\in\mathsf{S}}$ is μ-reversible. By construction, the diffusion measure P_s is unique for quasi-everywhere starting point s.

To construct labeled processes $\mathbf{X} = \{\mathbf{X}_t\}$ from unlabeled processes $\mathsf{X} = \{\mathsf{X}_t\}$ we introduce a subset $\mathsf{S}_{\text{s.i.}}$ of S. Let $\mathsf{S}_{\text{s.i.}}$ be the subset of S consisting of infinite-number of particles with no multiple points. Then by definition $\mathsf{S}_{\text{s.i.}}$ is given by

$$\mathsf{S}_{\text{s.i.}} = \{\mathsf{s} \in \mathsf{S}\,;\, \mathsf{s}(S) = \infty,\ \mathsf{s}(\{x\}) \le 1 \text{ for all } x \in S\}. \tag{2.24}$$

We assume that the subset H in (2.6) and the unlabeled diffusion $\{\mathsf{P}_\mathsf{s}\}_{\mathsf{s}\in\mathsf{S}}$ satisfies the following:

(A4) $\mathsf{P}_{\mathsf{s}}(\mathsf{X}_t \in \mathsf{S}_{\text{s.i.}}$ for all $t) = 1$ for q.e. $\mathsf{s} \in \mathsf{H}$.

The assumption **(A4)** means that the unlabeled process X consists of infinitely many particles and never collide each other. The equivalent condition of **(A4)** in terms of capacity is as follows.

(A4') $\mathrm{Cap}^{a,\mu}(\mathsf{S}_{\text{s.i.}}^c) = 0$.

Here $\mathrm{Cap}^{a,\mu}$ is the capacity associated with the Dirichlet form $(\mathcal{E}^{a,\mu}, \mathcal{D}^{a,\mu})$ on $L^2(\mathsf{S}, \mu)$ (see [9]). Recall that in (2.14) the local boundedness and uniform ellipticity of the coefficient a is assumed. Hence the assumption **(A4')** is equivalent to

(A4') $\mathrm{Cap}^{\mu}(\mathsf{S}_{\text{s.i.}}^c) = 0$.

Here and after, although Cap^{μ} depends on a in general, we suppress it from the notation.

Note that from **(A4')** we deduce that $\mu(\mathsf{S}_{\text{s.i.}}^c) = 0$.

Let $C([0,\infty); \mathsf{S}_{\text{s.i.}})$ be the set of $\mathsf{S}_{\text{s.i.}}$-valued paths. Each elements $\mathsf{X} \in C([0,\infty); \mathsf{S}_{\text{s.i.}})$ can be written as $\mathsf{X}_t = \sum_{i=1}^{\infty} \delta_{X_t^i}$, where $X^i = \{X_t^i\}_{t \in I^i} \in C(I^i; S)$ with (possibly random) interval I^i of the form $I^i = [0, b^i)$ or $I^i = (a^i, b^i)$ $(0 \le a^i < b^i \le \infty)$. The expression of the labeled path

$$\mathbf{X} = (X^i)_{i \in \mathbb{N}} \quad \text{where } X^i = \{X_t^i\} \in C(I^i, S), \tag{2.25}$$

is then unique up to labeling of each path X^i because of the assumption of $\mathsf{S}_{\text{s.i.}}$. We assume:

(A5) $\mathsf{P}_{\mathsf{s}}(I^i = [0,\infty)$ for all $i \in \mathbb{N}) = 1$ for all $\mathsf{s} \in \mathsf{H}$.

In Proposition 8.2 we will prove that **(A5)** follows from **(A2)**–**(A4)** and **(A5')** below.

(A5') The function a_0 in (2.14) is bounded from above $a_0(t) \le c_6$ for all $t \in [0,\infty)$ with some constant c_6, and there exists a $T > 0$ such that for each $R > 0$

$$\liminf_{r \to \infty} \mathcal{N}\Big(\frac{r}{\sqrt{(r+R)T}}\Big) \int_{|x| \le r+R} \rho^1(x) dx = 0. \tag{2.26}$$

Here $\mathcal{N}(t) = \int_t^{\infty} (1/\sqrt{2\pi}) e^{-|x|^2/2} dx$.

We remark that (2.26) is easy to check, and is satisfied if μ is a translation invariant random point field with 1-correlation function. In fact, $\rho^1 = $ const. in this case, which yields **(A5')**.

By **(A4)** and **(A5)** the diffusion measures P_{s} are supported on the set of non-explosion and non-collision paths. The advantage of this fact is that one can naturally lift each unlabeled path $\mathsf{X} \in C([0,\infty); \mathsf{H})$ to the labeled paths $\mathbf{X} \in C([0,\infty); S^{\mathbb{N}})$ by using a label map $\mathfrak{l} = (\mathfrak{l}^i)_{i \in \mathbb{N}}$. Indeed, by **(A4)** and **(A5)** we can construct the labeled process $\mathbf{X} = \{\mathbf{X}_t\} = \{(X_t^i)_{i \in \mathbb{N}}\}_{t \in [0,\infty)}$ for each $\mathsf{s} \in \mathsf{H}$ by $(X_0^i)_{i \in \mathbb{N}} = \mathfrak{l}(\mathsf{s})$ because each tagged particle in s can carry the initial label $\mathfrak{l}(\mathsf{s})$ forever. For a given label $\mathfrak{l}$ we write this correspondence by $\mathfrak{l}_{\text{path}}(\mathsf{X})$ and set $\mathbf{X} = \{\mathbf{X}_t\}_{t \in [0,\infty)}$ as follows.

$$\mathbf{X} = \mathfrak{l}_{\text{path}}(\mathsf{X}) \quad \text{with } \mathbf{X}_0 = \mathfrak{l}(\mathsf{s}). \tag{2.27}$$

We write $\mathbf{X} = (X^i)_{i\in\mathbb{N}}$ and $\mathfrak{l}_{\mathrm{path}}(\mathsf{X}) = (\mathfrak{l}^i_{\mathrm{path}}(\mathsf{X}))_{i\in\mathbb{N}}$. Then by construction $X^i = \mathfrak{l}^i_{\mathrm{path}}(\mathsf{X})$.

The next assumption is crucial for the passage from the unlabeled dynamics X to the labeled dynamics $\mathbf{X}$. To control $\mathbf{X} = (X^i)_{i\in\mathbb{N}}$ by $\mathsf{X} = \sum_{i\in\mathbb{N}} \delta_{X^i}$, we assume:

(A6) For each $r, T \in \mathbb{N}$

$$\mathsf{P}_\mu(\limsup_{i\to\infty} \mathsf{M}^i_{r,T}) = 0. \tag{2.28}$$

Here we set $\mathsf{M}^i_{r,T} = \{\mathsf{X} \in C([0,\infty);\mathsf{H})\,;\, |X^i_t| \le r \text{ for some } t \in [0,T]\}$.

One can easily see that the condition **(A6)** is independent of a particular choice of the label $\mathfrak{l}$ although the labeled dynamics $\mathbf{X} = (X^i)_{i\in\mathbb{N}} = \mathfrak{l}_{\mathrm{path}}(\mathsf{X})$ depends on the initial label $\mathfrak{l}$. We will give simple sufficient conditions of **(A6)** in Proposition 8.3 and Proposition 8.4.

To relate the labeled process $\mathbf{X}$ with the ISDE (2.5) and (2.6), we assume:

(A7) $\quad \mathsf{P}_{\mathsf{s}}(\mathsf{X}_t \in \mathsf{H} \text{ for all } t) = 1$ for all $\mathsf{s} \in \mathsf{H}$.

From this we deduce that X satisfies (2.6).

For a function f on $S \times \mathsf{S}$ we set $\check{f} = \check{f}(x_1, \ldots)$ by $\check{f}(x_1,\ldots) = f(x_1, \sum_{i\ge 2}\delta_{x_i})$. Hence by construction $\check{f}(x_1,x_2,x_3,\ldots)$ is symmetric in $(x_{m+1},x_{m+2},\ldots)$ for each $(x_1,,\ldots,x_m)$ for any $m \in \mathbb{N}$. With this notation, we set for σ and b in (2.5) that $\check{\sigma}(x,\mathbf{y}) = \sigma(x,\mathsf{y})$ and $\check{b}(x,\mathbf{y}) = b(x,\mathsf{y})$, where $\mathbf{y} = (y_i)$ and $\mathsf{y} = \sum_i \delta_{y_i}$.

For a label $\mathfrak{l}$ and $\mathsf{X} \in C([0,\infty);\mathsf{H})$ we set $\mathbf{X}$ as (2.27), and define the functions

$$\sigma^m_{\mathsf{X}} : [0,\infty)\times S \times S^{m-1} \to \mathbb{R}^{d^2} \quad\text{and}\quad b^m_{\mathsf{X}} : [0,\infty)\times S\times S^{m-1} \to \mathbb{R}^d \tag{2.29}$$

as

$$\sigma^m_{\mathsf{X}}(t,x,\mathbf{y}) = \sigma(x, \mathsf{y} + \mathsf{X}^{m*}_t), \quad b^m_{\mathsf{X}}(t,x,\mathbf{y}) = \check{b}(x,\mathsf{y}+\mathsf{X}^{m*}_t). \tag{2.30}$$

Here $\mathsf{X}^{m*}_t = \sum_{i=m+1}^\infty \delta_{X^i_t}$, $\mathbf{y} = (y_1,\ldots,y_{m-1})$, and $\mathsf{y} = \sum_{i=1}^{m-1}\delta_{y_i}$. Note that these functions are symmetric in $\mathbf{y}$ for each (t,x). Let $\mathbf{Y}^m = (Y^{m,i})_{i=1}^m$ and $\mathbf{Y}^{m,i\diamondsuit}_t = (Y^{m,j}_t)^m_{j\ne i}$. We assume:

(A8) For each $\mathsf{X} \in C([0,\infty);\mathsf{H})$ and $m \in \mathbb{N}$, the SDE of $\mathbf{Y}^m$ given by

$$dY^{m,i}_t = \sigma^m_{\mathsf{X}}(t, Y^{m,i}_t, \mathbf{Y}^{m,i\diamondsuit}_t)dB^i_t + b^m_{\mathsf{X}}(t, Y^{m,i}_t, \mathbf{Y}^{m,i\diamondsuit}_t)dt \quad (i=1,\ldots,m) \tag{2.31}$$

$$\mathbf{Y}^m_0 = \mathbf{s} \tag{2.32}$$

has a unique strong solution for any $\mathbf{s} = (s_i)_{i=1}^m \in S^m$ such that $\mathsf{s} + \mathsf{x}^*_m \in \mathsf{H}$. Here $\mathsf{s} = \sum_{i=1}^m \delta_{s_i}$ and $\mathsf{x}^*_m = \sum_{i=m+1}^\infty \delta_{x_i}$, where $\mathsf{x} = \sum_{i=1}^\infty \delta_{x_i} = \mathsf{X}_0$.

We remark that the SDE (2.31) is finite-dimensional. Hence one can apply the classical theory of SDEs directly. A feasible sufficient condition of **(A7)** and **(A8)** will be given in Subsection 8.2.

Under this condition we can take $\mathbf{s}$ in (2.32) to be any elements of

$$\{\mathbf{s} = (s_i)_{i=1}^m \in S^m \,;\, s_i \neq s_j \ (i \neq j), s_i \neq x_j \text{ for all } j \geq m+1\}$$

where $\mathsf{x} = \sum_{i=1}^{\infty} \delta_{x_i} = \mathsf{X}_0$.

We introduce the tail σ-field $\mathcal{T}(\mathsf{S})$ of S defined by

$$\mathcal{T}(\mathsf{S}) = \bigcap_{r=1}^{\infty} \sigma[\pi_{S_r^c}], \tag{2.33}$$

and assume that

(A9) $\mathcal{T}(\mathsf{S})$ is μ-trivial.

For given Radon measures m_1 and m_2 we write $m_1 \prec m_2$ if m_1 is absolutely continuous with respect to m_2. Let $\mathfrak{u}_{\text{path}}$ be the map on $C([0,\infty); S^{\mathbb{N}})$ defined by $\mathfrak{u}_{\text{path}}(\mathbf{X}) = \{\mathfrak{u}(\mathbf{X}_t)\}_{t\in[0,\infty)}$, where $\mathfrak{u}$ is the unlabel map. By definition, for $\mathbf{X}_t = (X_t^i)$ and $\mathsf{X}_t = \sum_i \delta_{X_t^i}$, we have

$$\mathfrak{u}_{\text{path}}(\mathbf{X}) = \mathsf{X}. \tag{2.34}$$

For $\mathbf{s}$ such that $\mathfrak{l}(\mathsf{s}) = \mathbf{s}$ and a probability measure $\mathbf{P}_{\mathbf{s}}$ on $C([0,\infty); S^{\mathbb{N}})$ such that $\mathbf{P}_{\mathbf{s}}(\mathbf{X}_0 = \mathbf{s}) = 1$, we set $\mathsf{P}_{\mathsf{s}} = \mathbf{P}_{\mathbf{s}} \circ \mathfrak{u}^{-1}$ and $\mathsf{P}_{\mu} = \int \mathsf{P}_{\mathsf{s}} \mu(d\mathsf{s})$. We now state our first main theorem.

Theorem 2.1. Assume (A1)–(A9) and let $\mathfrak{l}$ be a label. Then, for each $\mathbf{s} \in \mathfrak{l}(\mathsf{S}_0)$, there exists an $\mathsf{S}_0 \subset \mathsf{H}$ such that $\mu(\mathsf{S}_0) = 1$ and that the ISDE (2.5)–(2.7) has a strong solution $(\mathbf{X}, \mathbf{P}_{\mathbf{s}})$ satisfying

$$\mathsf{P}_{\mu} \circ \mathsf{X}_t^{-1} \prec \mu \quad \text{for all } t. \tag{2.35}$$

Furthermore, a strong uniqueness holds in the sense that a family of weak solutions $(\mathbf{X}, \mathbf{B}, \mathbf{P}_{\mathbf{s}})$ satisfying (2.35) is unique in law for $\mu^{\mathfrak{l}}$-a.s. $\mathbf{s} \in \mathsf{S}_0$ and becomes a strong solution, and in addition, for two strong solutions $(\mathbf{X}, \mathbf{B})$ and $(\mathbf{X}', \mathbf{B})$ staring at $\mathbf{s}$ defined on the same Brownian motion $\mathbf{B}$ satisfy, for $\mu^{\mathfrak{l}}$-a.s. $\mathbf{s} \in \mathsf{S}_0$,

$$\mathbf{X} = \mathbf{X}' \quad \text{for } P^{\mathbf{B}}\text{-a.s.}. \tag{2.36}$$

Remark 2.2. (1) Let $\mathsf{S}_0 = \mathfrak{u}(\mathbf{S}_0)$ and $\mathsf{s} = \mathfrak{u}(\mathbf{s})$, where $\mathfrak{u}(\mathbf{s}) = \sum_i \delta_{s_i}$ for $\mathbf{s} = (s_i)_{i\in\mathbb{N}}$ as before. The system of the associated unlabeled processes $(\mathsf{X}, \{\mathsf{P}_{\mathsf{s}}\}_{\mathsf{s}\in\mathsf{S}_0})$ is an S_0-valued, μ-reversible diffusion.
(2) Let $\mathbf{S}_0 = \mathfrak{l}(\mathsf{S}_0)$. Then Theorem 2.1 give a family of solution $(\mathbf{X}, \{\mathbf{P}_{\mathbf{s}}\}_{\mathbf{s}\in\mathbf{S}_0})$ by definition. We will see that we can extend the set $\mathbf{S}_0$ to $\hat{\mathbf{S}_0}$ such that the extended family of solution $(\mathbf{X}, \{\mathbf{P}_{\mathbf{s}}\}_{\mathbf{s}\in\hat{\mathbf{S}_0}})$ is a diffusion with state space $\hat{\mathbf{S}_0}$. We can do this by taking $\hat{\mathbf{S}_0}$ as a suitable subset of $\cup_{t\in[0,\infty)} \mathbf{X}_t(\mathbf{S}_0)$.
(3) The most prime assumption in Theorem 2.1 and Theorem 2.2 is (2.12) in **(A1)**. One may regard this as a differential equation concerning on random point fields μ. Hence, to some extent, we have solved the ISDE (2.5) through the geometric differential equation (2.12) in the infinite-dimensional space S.

Let $\mu_{\mathrm{Tail}}^{\mathsf{a}}$ be the regular conditional probability defined by

$$\mu_{\mathrm{Tail}}^{\mathsf{a}} = \mu(\cdot|\mathcal{T}(\mathsf{S}))(\mathsf{a}). \tag{2.37}$$

Since S is a Polish space, such a regular conditional probability exists, and satisfies the decomposition

$$\mu(\mathsf{A}) = \int_{\mathsf{S}} \mu_{\mathrm{Tail}}^{\mathsf{a}}(\mathsf{A})\mu(d\mathsf{a}). \tag{2.38}$$

By construction $\mu_{\mathrm{Tail}}^{\mathsf{a}}(\mathsf{A})$ are $\mathcal{T}(\mathsf{S})$-measurable functions in a for each $\mathsf{A} \in \mathcal{B}(\mathsf{S})$. Furthermore, we will prove in Lemma 7.1 that one can take a version of $\mu_{\mathrm{Tail}}^{\mathsf{a}}$ such that, for μ-a.s. $\mathsf{a} \in \mathsf{S}$,

$$\mu_{\mathrm{Tail}}^{\mathsf{a}}(\mathsf{A}) = 1_{\mathsf{A}}(\mathsf{a}) \quad \text{for all } \mathsf{A} \in \mathcal{T}(\mathsf{S}). \tag{2.39}$$

Let $\sim_{\mathrm{Tail}}$ be the equivalence relation such that $\mathsf{a} \sim_{\mathrm{Tail}} \mathsf{b}$ if and only if

$$1_{\mathsf{A}}(\mathsf{a}) = 1_{\mathsf{A}}(\mathsf{b}) \quad \text{for all } \mathsf{A} \in \mathcal{T}(\mathsf{S}). \tag{2.40}$$

From (2.39) we deduce that the set S_0 in Theorem 2.1 can be decomposed as a disjoint sum

$$\mathsf{S}_0 = \sum_{[\mathsf{a}] \in \mathsf{S}_0/\sim_{\mathrm{Tail}}} \mathsf{S}_0^{\mathsf{a}} \quad \text{such that} \quad \mu_{\mathrm{Tail}}^{\mathsf{a}}(\mathsf{S}_0^{\mathsf{a}}) = 1. \tag{2.41}$$

Assume **(A1)**–**(A3)**. Then $\mu_{\mathrm{Tail}}^{\mathsf{a}}$ satisfies **(A1)**–**(A3)** for μ-a.s. a. In fact, **(A1)** and **(A2)** are clear by definition and Fubini's theorem. **(A3)** follows from Fubini's theorem and Hölder's inequality. Hence by [28], for μ-a.s. a, there exists a subset $\mathsf{S}_0^{\mathsf{a}}$ such that $\mu_{\mathsf{a}}(\mathsf{S}_0^{\mathsf{a}}) = 1$ and that there exists an $\mathsf{S}_0^{\mathsf{a}}$-valued, $\mu_{\mathrm{Tail}}^{\mathsf{a}}$-reversible diffusion $(\mathsf{X}, \{\mathsf{P}_{\mathsf{s}}^{\mathsf{a}}\}_{\mathsf{s}\in\mathsf{S}_0^{\mathsf{a}}})$ associated with $(\mathcal{E}^{\mu_{\mathrm{Tail}}^{\mathsf{a}}}, \mathcal{D}^{\mu_{\mathrm{Tail}}^{\mathsf{a}}})$ on $L^2(\mathsf{S}, \mu_{\mathrm{Tail}}^{\mathsf{a}})$. Since $(\mathsf{X}, \{\mathsf{P}_{\mathsf{s}}^{\mathsf{a}}\}_{\mathsf{s}\in\mathsf{S}_0^{\mathsf{a}}})$ is $\mu_{\mathrm{Tail}}^{\mathsf{a}}$-reversible diffusion, $(\mathsf{X}, \{\mathsf{P}_{\mathsf{s}}^{\mathsf{a}}\}_{\mathsf{s}\in\mathsf{S}_0^{\mathsf{a}}})$ satisfies, for μ-a.s. a,

$$\mathsf{P}_{\mu_{\mathrm{Tail}}^{\mathsf{a}}}^{\mathsf{a}} \circ \mathsf{X}_t^{-1} \prec \mu_{\mathrm{Tail}}^{\mathsf{a}} \quad \text{for all } t \in [0, \infty). \tag{2.42}$$

We denote the conditions (A1)–(A8) for $\mu_{\mathrm{Tail}}^{\mathsf{a}}$ by (A1$_{\mathsf{a}}$)–(A8$_{\mathsf{a}}$) and relax the assumption **(A9)** of the tail triviality.

Theorem 2.2. Assume **(A1)**–**(A4)**, **(A5')**, **(A7)**, and **(A8)**. Assume **(A6$_{\mathsf{a}}$)** for μ-a.s. a. Let $\mathfrak{l}$ be a label. Then, for μ-a.s. a and $\mu^{\mathfrak{l}}$-a.s. $\mathbf{s}$, a strong solution $\mathbf{X}$ of the ISDE (2.5)–(2.7) whose unlabeled dynamics (X, P) satisfying (2.42) exists uniquely.

Remark 2.3. The uniqueness in Theorem 2.2 does not exclude the possibility that there exists a family of solutions $\mathbf{X}$ that does not satisfy (2.42), and that, in particular, a family of solutions whose distributions on the tail σ-field $\mathcal{T}(\mathsf{S})$ would change.

We give applications of these theorems. All the examples below fulfill the assumptions (A1)–(A8). In the rest of this section, we fix a label $\mathfrak{l}$. The parameter β are positive constants, denoting the inverse temperature.

Example 2.1 (Sine$_\beta$ random point fields/Dyson model in infinite dimensions)**.** Let $d = 1$ and $S = \mathbb{R}$. Let $\beta > 0$ and introduce the ISDE

$$dX_t^i = dB_t^i + \frac{\beta}{2} \lim_{r\to\infty} \sum_{|X_t^i - X_t^j|<r,\ j\neq i}^{\infty} \frac{1}{X_t^i - X_t^j} dt \quad (i \in \mathbb{Z}). \tag{2.43}$$

Clearly (2.43) is a special case of (1.1) with $\Phi(x) = 0$ and $\Psi(x, y) = -\beta \log |x - y|$.

Let $\mu_{\sin,\beta}$ be the sine$_\beta$ random point field [24, 10]. By definition $\mu_{\sin,2}$ is the random point field on $\mathbb{R}$ whose n-correlation function $\rho^n_{\sin,2}$ is given by

$$\rho^n_{\sin,2} = \det[\mathsf{K}_{\sin,2}(x_i, x_j)]_{i,j=1}^n. \tag{2.44}$$

Here $\mathsf{K}_{\sin,2}(x, y) = \sin \pi(x - y)/\pi(x - y)$ is the sine kernel. $\mu_{\sin,1}$ and $\mu_{\sin,4}$ are also defined by correlation functions given by quaternion determinants [24]. The random point fields $\mu_{\sin,\beta}$ $(\beta = 1, 2, 4)$ are solutions of the geometric differential equations (2.15) with $a = 1$ and

$$b(x, \mathsf{y}) = \frac{\beta}{2} \lim_{r\to\infty} \sum_{|x-y_i|<r} \frac{1}{x - y_i}. \tag{2.45}$$

Unlike Ginibre interacting Brownian motions, (2.45) is equivalent to

$$b(x, \mathsf{y}) = \frac{\beta}{2} \lim_{r\to\infty} \sum_{|y_i|<r} \frac{1}{x - y_i}. \tag{2.46}$$

We obtain the following.

Theorem 2.3. Let $\mathfrak{l}$ be a label. Suppose $\beta = 1, 2, 4$. Then the ISDE (2.43) has a unique strong solution satisfying (2.42) for $\mu^{\mathfrak{l}}_{\sin,\beta}$-a.s. $\mathbf{s}$.

When $\beta = 2$, the unique solution of the ISDE (2.43) is called Dyson model in infinite dimensions. Recently the random point fields $\mu_{\sin,\beta}$ are constructed for all $\beta > 0$ [49]. It would be plausible that Theorem 2.3 can be generalized to this case. We however do not pursue this here.

Example 2.2 (Airy random point fields)**.** Let $d = 1$ and $S = \mathbb{R}$. Let $\beta > 0$ and introduce the ISDE

$$dX_t^i = dB_t^i + \frac{\beta}{2} \lim_{r\to\infty} \Big\{ \Big(\sum_{j\neq i,\ |X_t^j|<r} \frac{1}{X_t^i - X_t^j} \Big) - \int_{|x|<r} \frac{\hat{\rho}(x)}{-x} dx \Big\} dt. \tag{2.47}$$

Here we set

$$\hat{\rho}(x) = \frac{1_{(-\infty,0)}(x)}{\pi} \sqrt{-x}. \tag{2.48}$$

Let $\mu_{\mathrm{Ai},\beta}$ be the Airy$_\beta$ random point field. By definition $\mu_{\mathrm{Ai},2}$ is a determinantal random point field whose n-point correlation function $\rho^n_{\mathrm{Ai},2}$ with respect to the Lebesgue measure is given by

$$\rho^n_{\mathrm{Ai},2}(\mathbf{x}_n) = \det[\mathsf{K}_{\mathrm{Ai},2}(x_i,x_j)]_{i,j=1}^n. \tag{2.49}$$

Here $\mathsf{K}_{\mathrm{Ai},2}(x_i,x_j)$ is a kernel given by

$$\mathsf{K}_{\mathrm{Ai},2}(x,y) = \frac{\mathrm{Ai}(x)\mathrm{Ai}'(y) - \mathrm{Ai}'(x)\mathrm{Ai}(y)}{x-y} \tag{2.50}$$

and Ai is the Airy function and Ai$'$ is its derivative. $\mu_{\mathrm{Ai},\beta}$ for $\beta = 1, 4$ are also given by a similar formula with quaternion determinant (see [24]). The following is proved in the co-paper [35].

Theorem 2.4 ([35])**.** Suppose $\beta = 1, 2, 4$. Then the ISDE (2.47) has a unique strong solution satisfying (2.42) for $\mu^{\mathfrak{l}}_{\mathrm{Ai},\beta}$-a.s. $\mathbf{s}$.

Example 2.3 (Ginibre random point field)**.** Let $d = 2$ and $S = \mathbb{R}^2$. Let $\beta = 2$. We introduce two different ISDEs of $\mathbf{X} = (X^i)_{i\in\mathbb{N}}$ on $(\mathbb{R}^2)^{\mathbb{N}}$ starting at $\mathbf{X}_0 = \mathbf{x}$:

$$dX^i_t = dB^i_t + \lim_{r\to\infty} \sum_{|X^i_t - X^j_t|<r,\, j\neq i} \frac{X^i_t - X^j_t}{|X^i_t - X^j_t|^2} dt \quad (i \in \mathbb{N}), \tag{2.51}$$

and

$$dX^i_t = dB^i_t - X^i_t dt + \lim_{r\to\infty} \sum_{|X^j_t|<r,\, j\neq i} \frac{X^i_t - X^j_t}{|X^i_t - X^j_t|^2} dt \quad (i \in \mathbb{N}). \tag{2.52}$$

Let μ_{Gin} be the Ginibre random point field. By definition μ_{Gin} is a random point field on $\mathbb{R}^2$ whose n-correlation functions w.r.t. the Lebesgue measure are given by

$$\rho^n_{\mathrm{Gin}}(x_1,\ldots,x_n) = \det[\mathsf{K}_{\mathrm{Gin}}(x_i,x_j)]_{i,j=1}^n, \tag{2.53}$$

where $\mathsf{K}_{\mathrm{Gin}}\colon \mathbb{R}^2\times\mathbb{R}^2 \to \mathbb{C}$ is the kernel defined by

$$\mathsf{K}_{\mathrm{Gin}}(x,y) = \pi^{-1} e^{-\frac{|x|^2}{2} - \frac{|y|^2}{2}} \cdot e^{x\bar{y}}. \tag{2.54}$$

Here we identify $\mathbb{R}^2$ as $\mathbb{C}$ by the obvious correspondence: $\mathbb{R}^2 \ni x = (x_1,x_2) \mapsto x_1 + \mathrm{i}x_2 \in \mathbb{C}$, and $\bar{y} = y_1 - \mathrm{i}y_2$ means the complex conjugate under this identification, where $\mathrm{i} = \sqrt{-1}$. It is known that $\mu_{\mathrm{Gin}}(\mathsf{S}_{\mathrm{s.i.}}) = 1$. Moreover, μ_{Gin} is translation and rotation invariant. μ_{Gin} is called the Ginibre random point field. We will prove that μ_{Gin} is a solution of the differential equation (2.15), and as a result we deduce the following from Theorem 2.1 and Theorem 2.2.

Theorem 2.5. Let $\mathfrak{l}$ be a label. Then the following holds.

(1) The ISDE (2.51) has a unique strong solution satisfying (2.42) for $\mu^{\mathfrak{l}}_{\mathrm{Gin}}$-a.s. $\mathbf{s}$.

(2) The ISDE (2.52) has a unique strong solution satisfying (2.42) for $\mu^{\mathfrak{l}}_{\mathrm{Gin}}$-a.s. $\mathbf{s}$.

(3) The solutions of the ISDEs (2.51) and (2.52) coincide for $\mu^{\mathfrak{l}}_{\mathrm{Gin}}$-a.s. $\mathbf{s}$.

理由：力学的 rigidity と（無限次元の）接空間の話を書く []

Example 2.4 (Ruelle's class potentials)**.** Let $S = \mathbb{R}^d$. Let $\Phi = 0$ and $\Psi(x, y) = \beta\Psi_0(x-y)$. Then (1.1) becomes

$$dX_t^i = dB_t^i - \frac{\beta}{2} \sum_{j=1, j\neq i}^{\infty} \nabla\Psi_0(X_t^i - X_t^j)dt \quad (i \in \mathbb{N}). \tag{2.55}$$

Assume that Ψ_0 is a Ruelle's class potential, smooth outside the origin. Namely, Ψ_0 is super stable and regular in the sense of Ruelle [42]. Here we say Ψ_0 is regular if there exists a positive decreasing function $\psi : \mathbb{R}^+ \to \mathbb{R}$ and R_0 such that

$$\begin{aligned} &\Psi_0(x) \geq -\psi(|x|) \quad \text{for all } x, \quad \Psi_0(x) \leq \psi(|x|) \quad \text{for all } |x| \geq R_0, \\ &\int_0^{\infty} \psi(t)\, t^{d-1} dt < \infty. \end{aligned} \tag{2.56}$$

For given Ruelle's class potentials Ψ_0, there exist translation invariant grand canonical Gibbs measures associated with Ψ_0 such that the m-point correlation function ρ^m with respect to the Lebesgue measure satisfies

$$\rho^m(x_1, \ldots, x_m) \leq c_7^m \tag{2.57}$$

for all $\mathbf{x}_m \in (\mathbb{R}^d)^m$ and $m \in \mathbb{N}$ (see [42]). Here c_7 is a positive constant.

Suppose that $d \geq 2$ or that $d = 1$ with Ψ_0 sufficiently repulsive at the origin [14] in the following sense: There exists a positive constant c_8 and a positive function $h : [0, \infty) \to [0, \infty]$ satisfying that

$$\int_{0 \leq t \leq c_8} h(t)^{-1} dt = \infty \tag{2.58}$$

and that

$$\rho^m(x_1, \ldots, x_m) \leq h(|x_i - x_j|) \quad \text{for all } x_i \neq x_j. \tag{2.59}$$

Then the associated translation invariant canonical Gibbs measures constructed in [42] satisfy **(A1)**–**(A8)**.

Theorem 2.6. Let $\beta > 0$ and Ψ_0 be Ruelle's class potential smooth outside the origin. Assume that $\nabla\Phi_0$ satisfies

$$|\nabla\Phi_0| \leq \psi(|x|) \quad \text{for all } |x| \geq R_0. \tag{2.60}$$

Let μ_{Ψ_0} be associated canonical Gibbs measures satisfying **(A2)**, **(A3)**, and **(A5')**. Assume $d \geq 2$ or $d = 1$ with (2.58) and (2.59). Then (2.55) has a unique strong solution satisfying (2.42) for μ_{Ψ_0}-a.s. **s**.

Remark 2.4. (1) Since the log potentials do not satisfy (2.56) for $d = 1, 2$, they are not Ruelle's class potentials. The associated random point fields such as Ginibre, sine$_\beta$, and Airy$_\beta$ ($\beta = 1, 2, 4$) are not canonical Gibbs measures in the sense that they do not satisfy (classical) DLR equations for these potentials.

(2) Suppose that μ are canonical Gibbs measures with locally bounded correlation functions $\{\rho^m\}_{m\in\mathbb{N}}$. Suppose that $d = 1$ and $\Psi_0(t) = -c_9|t|^{-\alpha}$ with a positive constant c_9. Then we see (2.58) and (2.59) for suitable h whenever $\alpha > 1$. Actually, (2.58) and (2.59) are finer than this. However, as far as associated quasi-Gibbs measures μ with locally bounded correlation functions exists, we can apply this result to μ. Note that, to make the potential Ψ_0 to be of Ruelle's class, we need $\alpha > 2$. We also remark that Inukai [14] proved that the pair of assumption (2.58) and (2.59) is a necessary and sufficient condition of the particles never collide for finite particle systems.

The next two examples are special cases of Example 2.4. Hence by Theorem 2.6 the associated ISDEs have unique strong solutions satisfying (2.42).

Example 2.5 (Lennard-Jones 6-12 potentials)**.** Let $d = 3$, $\beta > 0$, and $\Psi_{6,12}(x) = \{|x|^{-12} - |x|^{-6}\}$. The interaction $\Psi_{6,12}$ is called the Lennard-Jones 6-12 potential. The corresponding ISDE is:

$$dX_t^i = dB_t^i + \frac{\beta}{2}\sum_{j=1, j\neq i}^{\infty}\{\frac{12(X_t^i - X_t^j)}{|X_t^i - X_t^j|^{14}} - \frac{6(X_t^i - X_t^j)}{|X_t^i - X_t^j|^{8}}\}dt \quad (i \in \mathbb{N}). \tag{2.61}$$

Let $\mu_{6,12}$ be a canonical $\Psi_{6,12}$-Gibbs measure satisfying **(A3)** and **(A5')**. Note that such a random point field $\mu_{6.12}$ exists. Indeed, translation invariant, grand canonical Gibbs measures associated with $\Psi_{6,12}$ constructed in [42] fulfills all the conditions in Theorem 2.6.

Example 2.6. Let $\beta > 0$, $a > d \in \mathbb{N}$ and set $\Psi_a(x) = (\beta/a)|x|^{-a}$. The corresponding ISDE is:

$$dX_t^i = dB_t^i + \frac{\beta}{2}\sum_{j=1, j\neq i}^{\infty}\frac{X_t^i - X_t^j}{|X_t^i - X_t^j|^{a+2}}dt \quad (i \in \mathbb{N}). \tag{2.62}$$

At first glance the ISDE (2.62) resembles (2.51) and (2.43) because (2.62) corresponds to the case $a = 0$ in (2.51) and (2.43). The sums in the drift terms however converge absolutely unlike in (2.51) and (2.43). We emphasize that the structures of the dynamics given by the solutions of (2.62) are completely different from (2.51) and (2.43).

3 Solutions and IFC solutions of ISDEs

In this section we prove the existence of a unique, strong solution of ISDE (3.1)–(3.3) below. We will present a new formulation of the solution of the ISDE (3.1)–(3.3) (see Definition 3.1) and, by using this, derive the unique, strong solution in Theorem 3.4.

We fix $T \in \mathbb{N}$ and let $\|w\|_{C([0,T];S)} = \sup_{t\in[0,T]} |w(t)|$. Let $C_T(S^{\mathbb{N}}) = C([0,T];S^{\mathbb{N}})$ equipped with the Fréchet metric given by, for $\mathbf{w} = (w_n)_{n\in\mathbb{N}} \in C([0,T];S^{\mathbb{N}})$,

$$\mathrm{dist}(\mathbf{w},\mathbf{w}') = \sum_{n=1}^{\infty} 2^{-n} \min\{1, \|w_n - w'_n\|_{C([0,T];S)}\}.$$

Let $W_{T,\mathrm{sol}}$ be a Borel subset of $C_T(S^{\mathbb{N}})$. Let $\sigma^i, b^i : W_{T,\mathrm{sol}} \to C_T(S^{\mathbb{N}})$. Let $\mathbf{S}_0$ be a Borel subset of
説明 $S^{\mathbb{N}}$.

We consider a quadruplet $(\{\sigma^i\}, \{b^i\}, W_{T,\mathrm{sol}}, \mathbf{S}_0)$ and the ISDE on $S^{\mathbb{N}}$ of the form

$$dX_t^i = \sigma^i(\mathbf{X})_t dB_t^i + b^i(\mathbf{X})_t dt \quad (i \in \mathbb{N}) \tag{3.1}$$

$$\mathbf{X}_0 = \mathbf{s} = (s_i)_{i\in\mathbb{N}} \in \mathbf{S}_0 \tag{3.2}$$

$$\mathbf{X} \in W_{T,\mathrm{sol}}. \tag{3.3}$$

Here $\mathbf{X} = (X^i)_{i\in\mathbb{N}}$ and $\mathbf{B} = (B^i)_{i\in\mathbb{N}}$. $B^i = \{B_t^i\}$ $(i \in \mathbb{N})$ is a family of independent d-dimensional Brownian motions, and $\mathbf{B} = (B^i)_{i\in\mathbb{N}}$ is regarded as the $S^{\mathbb{N}}$-valued standard Brownian motion. We
説明 write $\mathbf{X} = \{\mathbf{X}_t\}_{t\in[0,T]}$, $\mathbf{X} = (X^i)_{i\in\mathbb{N}}$, and $X^i = \{X_t^i\}_{t\in[0,T]}$, and so on.

The set $\mathbf{S}_0$ is the collection of initial starting points of $\mathbf{X}$ and solutions, respectively. $W_{T,\mathrm{sol}}$ is that of solutions of the ISDE (3.1). We remark that, in infinite dimensions, SDEs does not make sense in the full space $C_T(S^{\mathbb{N}})$ usually. One see that the drift coefficients of the ISDEs of the present paper always converge in just a thin subset of the one labeled configuration space $S \times \mathsf{S}$. So it is natural to restrict the state space of the ISDE to a suitable subset $W_{T,\mathrm{sol}}$ of $C_T(S^{\mathbb{N}})$ with a suitable subset $\mathbf{S}_0$ of initial stating points $\mathbf{s}$.

We take $W_{T,\mathrm{sol}} = C([0,T];\mathsf{H})$, $\sigma^i(\mathbf{X})_t = \sigma(X_t^i, \mathsf{X}_t^{i\diamondsuit})$, and $b^i(\mathbf{X})_t = b(X_t^i, \mathsf{X}_t^{i\diamondsuit})$ in the ISDE (2.5)–(2.7) .

[1]

We take the attitude not to pose the explicit conditions of the coefficients σ^i and b^i to solve the ISDE (3.1), but to assume the existence of a weak solution (3.1) and the unique strong solutions of the associated infinite system of finite-dimensional SDEs instead. Our theorem will clarify a general structure on the relation between the unique existence of the strong solution of the ISDE (3.1) and the triviality of the tail σ-field of the labeled path space $\mathcal{T}_{\mathrm{path}}(S^{\mathbb{N}})$ defined by (3.16).

The final form of our general theorems (Theorem 2.1 and Theorem 2.2) are stated in terms of random point fields. We emphasize that there are many interesting RPFs satisfying the assumptions, such as the sine, Airy, Bessel, and Ginibre RPFs, and all canonical Gibbs measures with Ruelle's class potentials (with suitable smoothness of potentials that the associated ISDEs make sense).

[1] $W_{T,\mathrm{sol}}$ の位相をちゃんと考える。標準測度空間、および、F_∞ の極限移行が可能になる条件が必要

We fix an initial starting point $\mathbf{s}$ in (3.2), and assume:

(P1) The ISDE (3.1) has a weak solution $(\mathbf{X}, \mathbf{B})$.

Remark 3.1. More precisely, **(P1)** implies that there exists a pair of an $S^{\mathbb{N}}$-valued Brownian motion $\mathbf{B} = (B_t^i)_{i\in\mathbb{N}}$ and a $W_{T,\mathrm{sol}}$-valued random variable $\mathbf{X} = (X_t^i)_{i\in\mathbb{N}}$ satisfying the ISDE (3.1)–(3.3). Such a solution is often called a *solution*. In order to distinguish it from a strong solution, we will call it a weak solution. If $\mathbf{X}$ is a function of the Brownian motion $\mathbf{B}$, then $\mathbf{X}$ is called a strong solution. We will prove, under the assumption **(P2)**–**(P5)** below, $\mathbf{X}$ depends only on $\mathbf{B}$ and the initial condition $\mathbf{s}$.

For a labeled path $\mathbf{X} = (X^i)_{i\in\mathbb{N}} \in C_T(S^{\mathbb{N}})$ and $m \in \mathbb{N}$, we introduce the decomposition

$$\mathbf{X}^m = (X^1, \ldots, X^m) \in C_T(S^m), \quad \mathbf{X}^{m*} = (X^{m+1}, X^{m+2}, \ldots) \in C_T(S^{\mathbb{N}}).$$

Then clearly we have $\mathbf{X} = (\mathbf{X}^m, \mathbf{X}^{m*}) \in C_T(S^{\mathbb{N}})$.

For $\mathbf{s} \in \mathbf{S}_0$ we set $W_{T,\mathrm{sol}}^{\mathbf{s}} = \{\mathbf{X} \in W_{T,\mathrm{sol}}\,;\, \mathbf{X}_0 = \mathbf{s}\}$. For $\mathbf{s} \in \mathbf{S}_0$, $\mathbf{X} \in W_{T,\mathrm{sol}}^{\mathbf{s}}$, and $m \in \mathbb{N}$, we introduce the infinite system of finite-dimensional SDEs (3.4) and (3.5).

$$dY_t^{m,i} = \sigma^i((\mathbf{Y}^m, \mathbf{X}^{m*}))_t dB_t^i + b^i((\mathbf{Y}^m, \mathbf{X}^{m*}))_t dt \quad (i = 1, \ldots, m) \tag{3.4}$$

$$(\mathbf{Y}^m, \mathbf{X}^{m*}) \in W_{T,\mathrm{sol}}^{\mathbf{s}}. \tag{3.5}$$

Here $\mathbf{Y}^m = (Y^{m,1}, \ldots, Y^{m,m})$ is an unknown process, $\mathbf{X}^{m*}$ is interpreted as a part of the coefficients of the SDE (3.4). By definition we have

$$(\mathbf{Y}^m, \mathbf{X}^{m*}) = (Y^{m,1}, \ldots, Y^{m,m}, X^{m+1}, X^{m+2}, \ldots) \in C_T(S^{\mathbb{N}}). \tag{3.6}$$

From (3.5) we deduce that $(\mathbf{Y}^m, \mathbf{X}^{m*})_0 = \mathbf{s}$, and hence $\mathbf{Y}^m$ is a solution of (3.4) starting at $\mathbf{s}^m = (s_1, \ldots, s_m)$.

We fix $\mathbf{s} \in \mathbf{S}_0$ and assume:

(P2) For each $\mathbf{X} \in W_{T,\mathrm{sol}}$ such that $\mathbf{X}_0 = \mathbf{s}$, the SDE (3.4) and (3.5) has a unique, strong solution $\mathbf{Y}^m$ for each $m \in \mathbb{N}$.

Remark 3.2. (1) In **(P2)**, the notion of strong solutions means that $\mathbf{Y}^m$ is a function of given Brownian motion $(B_t^i)_{i=1,\ldots,m}$ and the first m-components $(s_1, \ldots, s_m)$ of given initial condition $\mathbf{s}$. We interpret $\mathbf{X}^{m*}$ as a part of coefficients. Hence $\mathbf{Y}^m$ is a functional of $\mathbf{s}$, $\mathbf{B}$, and $\mathbf{X}^{m*}$.
(2) The standard theory of SDEs asserts that the pathwise uniqueness implies the uniqueness in law (see [13, 41, **?**]).

We will prove the unique existence of strong solutions of the ISDE (3.1)–(3.3). For this we prepare a new formulation of a solution of the ISDE (3.1). We will interpret the ISDE as a consistent family of finite-dimensional SDEs with trivial tail.

Let $\mathbf{0} = (0, \dots) \in S^{\mathbb{N}}$ and set $C_T^{\mathbf{0}}(S^{\mathbb{N}}) = \{\mathbf{X} \in C_T(S^{\mathbb{N}})\,;\, \mathbf{X}_0 = \mathbf{0}\}$. [2] Let $\mathcal{S}$ be a Polish space defined by

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$$\mathcal{S} = \mathbf{S}_0 \times W_{T,\mathrm{sol}} \times C_T^{\mathbf{0}}(S^{\mathbb{N}}) \subset S^{\mathbb{N}} \times W_{T,\mathrm{sol}} \times C_T^{\mathbf{0}}(S^{\mathbb{N}}). \tag{3.8}$$

Define the map $F^m : \mathcal{S} \to W_{T,\mathrm{sol}}$ by

:31e

$$F^m(\mathbf{s}, \mathbf{X}, \mathbf{B}) = \{(Y_t^{m,1}, \dots, Y_t^{m,m}, X_t^{m+1}, X_t^{m+2}, \dots)\}_{0 \le t \le T}. \tag{3.9}$$

Here $\mathbf{Y}^m = (Y_t^{m,1}, \dots, Y_t^{m,m})$ is the unique, strong solution given by (3.4) and (3.5). We deduce from **(P2)** that the map F^m is well defined. For fixed $(\mathbf{s}, \mathbf{B})$, $F^m(\mathbf{s}, \cdot, \mathbf{B})$ defines a map from $W_{T,\mathrm{sol}}$ to $W_{T,\mathrm{sol}}$. Hence the composition $F^m \circ \cdots \circ F^1$ form a flow on $W_{T,\mathrm{sol}}$. It is clear that

[] 後で考える $F^m \circ \cdots \circ F^1 = F^m$. Moreover, $F^i \circ F^j \neq F^j \circ F^i$ in general.

(3.10) For a probability measure $\bar{P}_{\mathbf{s}}$ on $\mathcal{S}$ and a subset $\mathbf{S}_0 \subset S^{\mathbb{N}}$ we say

:31g

$$F^{\infty}(\mathbf{s}, \mathbf{X}, \mathbf{B}) = \lim_{m \to \infty} F^m(\mathbf{s}, \mathbf{X}, \mathbf{B}) \text{ in } W_{T,\mathrm{sol}} \text{ on } \mathbf{S}_0 \tag{3.10}$$

if $F^{\infty}(\mathbf{s}, \mathbf{X}, \mathbf{B}) \in W_{T,\mathrm{sol}}$ and, for all $\mathbf{s} \in \mathbf{S}_0$ and $\bar{P}_{\mathbf{s}}$-a.s. $(\mathbf{X}, \mathbf{B})$, the following converges in $C_T(S^{\mathbb{N}})$:

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$$\lim_{m \to \infty} F^m(\mathbf{s}, \mathbf{X}, \mathbf{B}) = F^{\infty}(\mathbf{s}, \mathbf{X}, \mathbf{B}), \tag{3.11}$$

:31i

$$\lim_{m \to \infty} \int_0^{\cdot} \sigma(F^m(\mathbf{s}, \mathbf{X}, \mathbf{B}))_u d\mathbf{B}_u = \int_0^{\cdot} \sigma(F^{\infty}(\mathbf{s}, \mathbf{X}, \mathbf{B}))_u d\mathbf{B}_u, \tag{3.12}$$

:31j

$$\lim_{m \to \infty} \int_0^{\cdot} b(F^m(\mathbf{s}, \mathbf{X}, \mathbf{B}))_u du = \int_0^{\cdot} b(F^{\infty}(\mathbf{s}, \mathbf{X}, \mathbf{B}))_u du. \tag{3.13}$$

Here $\int_0^{\cdot} \sigma(F^m(\mathbf{s}, \mathbf{X}, \mathbf{B}))_u d\mathbf{B}_u = (\int_0^{\cdot} \sigma^i(F^m(\mathbf{s}, \mathbf{X}, \mathbf{B}))_u dB_u^i)_{i \in \mathbb{N}} \in C_T(S^{\mathbb{N}})$, and other integrals in (3.12) and (3.13) are defined in a similar fashion.

Let P_{Br}^{∞} be the distribution of the standard Brownian motion in $S^{\mathbb{N}}$. By definition P_{Br}^{∞} is the infinite product of the Wiener measure on S with mean zero and unit variance matrix.

With these preparation we introduce a new notion of solutions of ISDEs. This is an equivalent notion of the ISDEs (3.1)–(3.3) in terms of *an infinite system of finite-dimensional SDEs with consistency* (IFC).

dfn:31

Definition 3.1. (1) A probability measure $\bar{P}_{\mathbf{s}}$ on $C_T(S^{\mathbb{N}}) \times C_T^{\mathbf{0}}(S^{\mathbb{N}})$ is called an IFC solution of (3.1)–(3.3) if $\bar{P}_{\mathbf{s}}$ satisfies (3.10), (3.14), and (3.15).

:31x

$$\bar{P}_{\mathbf{s}}(W_{T,\mathrm{sol}}^{\mathbf{s}} \times C_T^{\mathbf{0}}(S^{\mathbb{N}})) = 1, \quad \text{where } W_{T,\mathrm{sol}}^{\mathbf{s}} = \{\mathbf{X} \in W_{T,\mathrm{sol}}\,;\, \mathbf{X}_0 = \mathbf{s}\}, \tag{3.14}$$

:31y

$$\bar{P}_{\mathbf{s}}(\mathbf{B} \in \cdot) = P_{\mathrm{Br}}^{\infty}. \tag{3.15}$$

[2]Let $\mathcal{S}$ be a Borel subset of $S^{\mathbb{N}} \times C_T^{\mathbf{0}}(S^{\mathbb{N}}) \times W_{T,\mathrm{sol}}$ defined by

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$$\mathcal{S} = \bigcup_{\mathbf{s} \in \mathbf{S}_0} \{\mathbf{s}\} \times W_{T,\mathrm{sol}} \times C_T^{\mathbf{0}}(S^{\mathbb{N}}) \tag{3.7}$$

(2) We say $\{\bar{P}_{\mathbf{s}}\}_{\mathbf{s}\in\mathbf{S}_0}$ IFC solutions of (3.1)–(3.3) on $\mathbf{S}_0$ if $\bar{P}_{\mathbf{s}}$ are IFC solutions of (3.1)–(3.3) for each $\mathbf{s} \in \mathbf{S}_0$.

We will use the notion of IFC solutions to prove the main theorem in this section (Theorem 3.4). We will assume the tail triviality for the distribution of the weak solution in **(P1)**. This assumption is a key to the construction of strong solution of the ISDE (3.1). We introduce the notion related to the tail triviality of the labeled path space.

Let $\mathcal{F}$ be a σ-field and P be a probability measure defined on $\mathcal{F}$. We call $\mathcal{F}$ to be P-trivial (trivial with respect to P) if $P(A) \in \{0, 1\}$ for all $A \in \mathcal{F}$. We say that P is tail trivial if P is trivial on the tail σ-field (if the tail σ-field is suitably defined). For a sub σ-field $\mathcal{G}$ of $\mathcal{F}$, we write $\mathcal{F} = \mathcal{G}$ P-a.s. if for each $B \in \mathcal{F}$, there exist $A_1, A_2 \in \mathcal{G}$ such that $A_1 \subset B \subset A_2$ and that $P(A_2 \backslash A_1) = 0$. $C_T(S^{\mathbb{N}}) C_T^{\mathbf{s}}(S^{\mathbb{N}})$ どちらが良いか

For a measurable space $(\Omega, \mathcal{F})$ and a family of probability measures $\{P_\lambda\}_\Lambda$ on it, we say $(\Omega, \mathcal{F})$ is countably determined under $\{P_\lambda\}_\Lambda$ if there exists a family $\{A_n\}_{n\in\mathbb{N}}$ of countable elements in $\mathcal{F}$ such that $P_\lambda = P_{\lambda'}$ if and only if $P_\lambda(A_n) = P_{\lambda'}(A_n)$ for all $n \in \mathbb{N}$. We call $(\Omega, \mathcal{F})$ countably determined if it is countably determined under any family of probability measures. It is known that a Polish space equipped with Borel σ-field is countably determined under any family of probability measures. On the other hand, a Polish space with a sub σ-field $\mathcal{G}$ of the Borel σ-field is not necessary countably determined.

Let $\mathcal{T}_{\mathrm{path}}(S^{\mathbb{N}})$ be the tail σ-field of $C_T(S^{\mathbb{N}})$ defined by

$$\mathcal{T}_{\mathrm{path}}(S^{\mathbb{N}}) = \bigcap_{m=1}^{\infty} \sigma[\mathbf{X}^{m*}]. \tag{3.16}$$

We remark that it is not clear (or not true) that the tail σ-field $\mathcal{T}_{\mathrm{path}}(S^{\mathbb{N}})$ is countably determined.

Lemma 3.1. Assume **(P1)** and **(P2)**. Let $\bar{P}_{\mathbf{s}}$ be the distribution of the solution of the ISDE (3.1)–(3.3) in **(P1)**. Then the following holds.

(1) The sequence of maps $\{F^m\}_{m\in\mathbb{N}}$ is consistent in the sense that, for $\bar{P}_{\mathbf{s}}$-a.s. $(\mathbf{X}, \mathbf{B})$,

$$F^m(\mathbf{s}, \mathbf{X}, \mathbf{B})^{[m]} = F^{m+n}(\mathbf{s}, \mathbf{X}, \mathbf{B})^{[m]} \quad \text{for all } n \in \mathbb{N}, \tag{3.17}$$

where $\mathbf{X}^{[m]}$ denotes the first m components of $\mathbf{X}$.

(2) The probability measure $\bar{P}_{\mathbf{s}}$ is an IFC solution. Furthermore, $(\mathbf{X}, \mathbf{B})$ is a fixed point of F^∞ in (3.10) in the sense that

$$(F^\infty(\mathbf{s}, \mathbf{X}, \mathbf{B}), \mathbf{B}) = (\mathbf{X}, \mathbf{B}) \quad \text{for } \bar{P}_{\mathbf{s}}\text{-a.s. } (\mathbf{X}, \mathbf{B}). \tag{3.18}$$

(3) The map $F^\infty(\mathbf{s}, \cdot, \cdot)$ is $\mathcal{T}_{\mathrm{path}}(S^{\mathbb{N}}) \times \mathcal{B}(C_T^{\mathbf{0}}(S^{\mathbb{N}}))$-measurable. Moreover, if **(P1)** and **(P2)** hold for all $\mathbf{s} \in \mathbf{S}_0$, then the map F^∞ defined on $\mathcal{S}$ is $\mathcal{B}(\mathbf{S}_0) \times \mathcal{T}_{\mathrm{path}}(S^{\mathbb{N}}) \times \mathcal{B}(C_T^{\mathbf{0}}(S^{\mathbb{N}}))$-measurable.

Proof. The consistency (3.17) follows from **(P1)** and **(P2)**. Hence we obtain (1).

(3.10) is immediate from the consistency (3.17). (3.14) and (3.15) are obvious because $\bar{P}_{\mathbf{s}}$ is the distribution of the solution $(\mathbf{X},\mathbf{B})$ of (3.1)–(3.3). Hence $\bar{P}_{\mathbf{s}}$ is an IFC solution. (3.18) is clear because $(\mathbf{X},\mathbf{B})$ is a solution of (3.1)–(3.3), which yields (2).

Let $\mathcal{T}_{\mathrm{path}}^{m}(S^{\mathbb{N}}) = \cap_{n=m+1}^{\infty}\sigma[X^n]$. Then we deduce that $\mathcal{T}_{\mathrm{path}}(S^{\mathbb{N}}) = \cap_{m=1}^{\infty}\mathcal{T}_{\mathrm{path}}^{m}(S^{\mathbb{N}})$ and that F^m is $\mathcal{B}(\mathbf{S}_0)\times\mathcal{T}_{\mathrm{path}}^{m}(S^{\mathbb{N}})\times\mathcal{B}(C_T^{\mathbf{0}}(S^{\mathbb{N}}))$-measurable. Hence F^n are $\mathcal{T}_{\mathrm{path}}^{m}(S^{\mathbb{N}})$-measurable for all $m\le n\in\mathbb{N}$. Combining this with (3.10), we obtain the second claim of (3). The proof of the first claim of (3) is similar to the second. □

The next lemma reveals the relation between solutions and IFC solutions.

Lemma 3.2. Assume **(P2)**. Assume that $\bar{P}_{\mathbf{s}}$ is an IFC solution of the ISDE (3.1)–(3.3) and set $\mathbf{Y}=F^{\infty}(\mathbf{s},\mathbf{X},\mathbf{B})$. Then $(\mathbf{Y},\mathbf{B})$ under $\bar{P}_{\mathbf{s}}$ is a weak solution of the ISDE (3.1)–(3.3).

Proof. We write $\mathbf{Y}=(Y^n)_{n\in\mathbb{N}}$. From **(P2)** and the definition of IFC solutions we see that

$$Y_t^i - Y_0^i = \lim_{m\to\infty}\int_0^t \sigma^i(\mathbf{Y}^{[m]}+\mathbf{X}^{[m*]})_u dB_u^i + \lim_{m\to\infty}\int_0^t b^i(\mathbf{Y}^{[m]}+\mathbf{X}^{[m*]})_u du \tag{3.19}$$
$$= \int_0^t \sigma^i(\mathbf{Y})_u dB_u^i + \int_0^t b^i(\mathbf{Y})_u du.$$

Here we used the fact that $\mathbf{X}$ is $W_{T,\mathrm{sol}}$-valued in the second line. From (3.19), we deduce that $(\mathbf{Y},\mathbf{B})$ under $\bar{P}_{\mathbf{s}}$ is a solution of the ISDE (3.1)–(3.3). □

For a probability measure $\bar{P}_{\mathbf{s}}$ on $C_T(S^{\mathbb{N}})\times C_T^{\mathbf{0}}(S^{\mathbb{N}})$ we denote by $\bar{P}_{\mathbf{s},\mathbf{B}}$ the regular conditional probability

$$\bar{P}_{\mathbf{s},\mathbf{B}} = \bar{P}_{\mathbf{s}}(\mathbf{X}\in\cdot|\mathbf{B}). \tag{3.20}$$

Here we write $(\mathbf{X},\mathbf{B})\in C_T(S^{\mathbb{N}})\times C_T^{\mathbf{0}}(S^{\mathbb{N}})$.

Let $\mathcal{T}_{\mathrm{path}}(S^{\mathbb{N}})$ be as in (3.16). For a probability measure P on $C_T(S^{\mathbb{N}})$ we set

$$\mathcal{T}_{\mathrm{path}}^{[1]}(S^{\mathbb{N}};P) = \{\mathbf{A}\in\mathcal{T}_{\mathrm{path}}(S^{\mathbb{N}})\,;\,P(\mathbf{A})=1\}. \tag{3.21}$$

The next theorem reveals the relation between strong solutions and the tail triviality of the labeled path spaces $C_T(S^{\mathbb{N}})$. We fix $\mathbf{s}\in\mathbf{S}_0$ in the next theorem.

Theorem 3.3. Let $\bar{P}_{\mathbf{s}}$ be an IFC solution of the ISDE (3.1)–(3.3). Then the following holds.

(1) Set $\mathbf{Y}=F^{\infty}(\mathbf{s},\mathbf{X},\mathbf{B})$. Then $(\mathbf{Y},\mathbf{B})$ under $\bar{P}_{\mathbf{s}}$ is a strong solution of the ISDE (3.1)–(3.3) if and only if $\mathcal{T}_{\mathrm{path}}(S^{\mathbb{N}})$ is $\bar{P}_{\mathbf{s},\mathbf{B}}$-trivial for P_{Br}^{∞}-a.s. $\mathbf{B}$.

(2) Let $\mathbf{X}$ and $\mathbf{X}'$ be strong solutions of the ISDE (3.1)–(3.3) defined on the same Brownian motion $\mathbf{B}$. Let $\bar{P}_{\mathbf{s}}$ and $\bar{P}_{\mathbf{s}}'$ be the distributions of $(\mathbf{X},\mathbf{B})$ and $(\mathbf{X}',\mathbf{B})$, respectively. Then $\mathbf{X}=\mathbf{X}'$ for a.s. $\mathbf{B}$ if and only if $\mathcal{T}_{\mathrm{path}}^{[1]}(S^{\mathbb{N}};\bar{P}_{\mathbf{s},\mathbf{B}}) = \mathcal{T}_{\mathrm{path}}^{[1]}(S^{\mathbb{N}};\bar{P}_{\mathbf{s},\mathbf{B}}')$ for a.s. $\mathbf{B}$.

(3) Assume **(P2)**. Then, a strong solution of the ISDE (3.1)–(3.3) is unique if and only if, for P_{Br}^{∞}-a.s. $\mathbf{B}$, the set $\mathcal{T}_{\mathrm{path}}^{[1]}(S^{\mathbb{N}};\bar{P}_{\mathbf{s},\mathbf{B}})$ is independent of an IFC solution $\bar{P}_{\mathbf{s}}$.

Proof. Recall that $\bar{P}_{\mathbf{s},\mathbf{B}}$ is the distribution of $\mathbf{Y}$ for given $(\mathbf{s},\mathbf{B})$. Suppose that $(\mathbf{Y},\mathbf{B})$ under $\bar{P}_{\mathbf{s}}$ is a strong solution of the ISDE (3.1)–(3.3). Then $\mathbf{Y}$ is a function of $(\mathbf{s},\mathbf{B})$ by definition. Hence $\mathcal{T}_{\mathrm{path}}(S^{\mathbb{N}})$ is $\bar{P}_{\mathbf{s},\mathbf{B}}$-trivial for P_{Br}^{∞}-a.s. $\mathbf{B}$.

Conversely, if $\mathcal{T}_{\mathrm{path}}(S^{\mathbb{N}})$ is $\bar{P}_{\mathbf{s},\mathbf{B}}$-trivial for P_{Br}^{∞}-a.s. $\mathbf{B}$, then $\mathbf{Y}$ under $\bar{P}_{\mathbf{s},\mathbf{B}}$ is a constant. Hence $\mathbf{Y}$ under $\bar{P}_{\mathbf{s}}$ becomes a function of $\mathbf{B}$, which implies $(\mathbf{Y},\mathbf{B})$ under $\bar{P}_{\mathbf{s}}$ is a strong solution. We have thus obtained (1).

We next prove (2). By assumption $\mathbf{X}$ and $\mathbf{X}'$ are strong solutions with the same Brownian motions. Hence, under $\mathbf{P}_{\mathbf{s}}$ and $\mathbf{P}_{\mathbf{s}}'$,

$$F^{\infty}(\mathbf{s},\mathbf{X},\mathbf{B})=\mathbf{X} \text{ and } F^{\infty}(\mathbf{s},\mathbf{X}',\mathbf{B})=\mathbf{X}', \quad \text{respectively.} \tag{3.22}$$

From (3.22) combined with (1), we deduce that $\mathcal{T}_{\mathrm{path}}(S^{\mathbb{N}})$ is trivial under both of $\bar{P}_{\mathbf{s},\mathbf{B}}$ and $\bar{P}_{\mathbf{s},\mathbf{B}}'$.

We now suppose that

$$\mathcal{T}_{\mathrm{path}}^{[1]}(S^{\mathbb{N}};\bar{P}_{\mathbf{s},\mathbf{B}})=\mathcal{T}_{\mathrm{path}}^{[1]}(S^{\mathbb{N}};\bar{P}_{\mathbf{s},\mathbf{B}}') \text{ for a.s. } \mathbf{B}. \tag{3.23}$$

Since $F^{\infty}(\mathbf{s},\cdot,\mathbf{B})$ are $\mathcal{T}_{\mathrm{path}}(S^{\mathbb{N}})$-measurable, we see that $\mathbf{X}=\mathbf{X}'$ for a.s. $\mathbf{B}$ from (3.22), (3.23), and the triviality of $\mathcal{T}_{\mathrm{path}}(S^{\mathbb{N}})$ under $\bar{P}_{\mathbf{s},\mathbf{B}}$ and $\bar{P}_{\mathbf{s},\mathbf{B}}'$.

The converse statement in (2) is obvious. Indeed, if $\mathbf{X}=\mathbf{X}'$ for a.s. $\mathbf{B}$, then the image measures $\bar{P}_{\mathbf{s},\mathbf{B}}$ and $\bar{P}_{\mathbf{s},\mathbf{B}}'$ defined in (3.20) are the same. We have thus obtained (2).

We proceed to (3). Suppose that, for P_{Br}^{∞}-a.s. $\mathbf{B}$, $\mathcal{T}_{\mathrm{path}}^{[1]}(S^{\mathbb{N}};\bar{P}_{\mathbf{s},\mathbf{B}})$ is independent of an IFC solution $\bar{P}_{\mathbf{s}}$. Let $\mathbf{X}$ and $\mathbf{X}'$ be strong solutions of the ISDE (3.1)–(3.3) defined on same Brownian motion $\mathbf{B}$. Let $\bar{P}_{\mathbf{s}}$ and $\bar{P}_{\mathbf{s}}'$ be the distributions of $(\mathbf{X},\mathbf{B})$ and $(\mathbf{X}',\mathbf{B})$. Then, from Lemma 3.1 (2), we see that $\bar{P}_{\mathbf{s}}$ and $\bar{P}_{\mathbf{s}}'$ are IFC solutions. Hence we deduce (3.23) by assumption. Combining this with (2), we obtain the uniqueness of the strong solution of the ISDE (3.1)–(3.3). The converse statement is obvious. □

In the rest of this section, we assume **(P1)** and **(P2)**, and denote by $\bar{P}_{\mathbf{s}}$ the distribution of the solution of the ISDE (3.1)–(3.3) given by **(P1)**. From **(P2)** and Lemma 3.1 (2), $\bar{P}_{\mathbf{s}}$ is an IFC solution of the ISDE (3.1)–(3.3). Let $\mathbf{s}\in\mathbf{S}_0$ be fixed as in **(P1)**. We assume:

(P3) $\mathcal{T}_{\mathrm{path}}(S^{\mathbb{N}})$ is $\bar{P}_{\mathbf{s},\mathbf{B}}$-trivial for P_{Br}^{∞}-a.s. $\mathbf{B}$.
(P4) The set $\mathcal{T}_{\mathrm{path}}^{[1]}(S^{\mathbb{N}};\bar{P}_{\mathbf{s},\mathbf{B}})$ is independent of P_{Br}^{∞}-a.s. $\mathbf{B}$.
(P5) The set $\mathcal{T}_{\mathrm{path}}^{[1]}(S^{\mathbb{N}};\bar{P}_{\mathbf{s},\mathbf{B}})$ is independent of the distribution $\bar{P}_{\mathbf{s}}$ and P_{Br}^{∞}-a.s. $\mathbf{B}$.

From Theorem 3.3 we immediately obtain the following.

Theorem 3.4. (1) Assume **(P1)**–**(P3)**. Then the ISDE (3.1)–(3.3) has a strong solution.
(2) Assume **(P1)**–**(P4)**. Then a strong solution of the ISDE (3.1)–(3.3) with the distribution $\bar{P}_{\mathbf{s}}$ is unique.
(3) Assume **(P1)**–**(P5)**. Then the ISDE (3.1)–(3.3) has a unique, strong solution.

Proof. The claims (1), (2), and (3) follow from (1), (2), and (3) of Theorem 3.3, respectively. □

4 Derivation of the tail triviality of $C_T(S^{\mathbb{N}})$ from the cylindrical tail triviality.

Let $\mathcal{T}_{\mathrm{path}}(S^{\mathbb{N}})$ be the tail σ-field of the labeled path space $C_T(S^{\mathbb{N}})$ introduced in (3.16). As we see in Theorem 3.4, the crucial step to construct strong solutions is to prove the triviality of $\mathcal{T}_{\mathrm{path}}(S^{\mathbb{N}})$. This problem is, however, quite hard in general because $C_T(S^{\mathbb{N}})$ is a very huge space and the tail σ-field $\mathcal{T}_{\mathrm{path}}(S^{\mathbb{N}})$ is topologically wild. Hence we give its sufficient conditions in this section. For this we introduce the notion of the *cylindrical* tail σ-field $\tilde{\mathcal{T}}_{\mathrm{path}}(S^{\mathbb{N}})$ and deduce the triviality of $\mathcal{T}_{\mathrm{path}}(S^{\mathbb{N}})$ from that of $\tilde{\mathcal{T}}_{\mathrm{path}}(S^{\mathbb{N}})$.

Let $\tilde{\mathcal{T}}_{\mathrm{path}}(S^{\mathbb{N}})$ be the cylindrical tail σ-field of the labeled path space $C_T(S^{\mathbb{N}})$ defined by

$$\tilde{\mathcal{T}}_{\mathrm{path}}(S^{\mathbb{N}}) = \bigvee_{\mathbf{t}\in\mathbf{T}} \bigcap_{n=1}^{\infty} \sigma[\mathbf{X}_{\mathbf{t}}^{n*}]. \tag{4.1}$$

Here $\mathbf{T} = \{\mathbf{t} = (t_1, \ldots, t_m)\,;\, t_i \in [0,T], m \in \mathbb{N}\}$ and $\mathbf{X}_{\mathbf{t}}^{n*} = (\mathbf{X}_{t_1}^{n*}, \ldots, \mathbf{X}_{t_m}^{n*})$. By construction

$$\tilde{\mathcal{T}}_{\mathrm{path}}(S^{\mathbb{N}}) \subset \mathcal{T}_{\mathrm{path}}(S^{\mathbb{N}}). \tag{4.2}$$

This inclusion (4.2) suggests the strategy to deduce the triviality of $\mathcal{T}_{\mathrm{path}}(S^{\mathbb{N}})$ from that of $\tilde{\mathcal{T}}_{\mathrm{path}}(S^{\mathbb{N}})$, and after that to prove the triviality of $\tilde{\mathcal{T}}_{\mathrm{path}}(S^{\mathbb{N}})$. We will do the former in Section 4 and the latter in Section 5.

We say a probability measure P on the labeled path space $C_T(S^{\mathbb{N}})$ is cylindrically tail trivial if $P(A) \in \{0,1\}$ for all $A \in \tilde{\mathcal{T}}_{\mathrm{path}}(S^{\mathbb{N}})$. Our task in the present section is thus the derivation of the triviality of the labeled path space from the cylindrical tail triviality.

Let $\mathcal{S}$ and $\mathbf{S}_0$ be as in Section 3. In the present section we fix the initial staring point $\mathbf{s} \in \mathbf{S}_0$. For a probability measure $\bar{P}_{\mathbf{s}}$ on $\mathcal{S}$ we denote by $\mathbf{P}_{\mathbf{s}}$ the distribution of $\mathbf{X}$:

$$\mathbf{P}_{\mathbf{s}} = \bar{P}_{\mathbf{s}}(\mathbf{X} \in \cdot). \tag{4.3}$$

Recall the notion of the IFC solution introduced by Definition 3.1. Then, for each $\mathbf{s} \in \mathbf{S}_0$, we assume the following:

(T1) $\bar{P}_{\mathbf{s}}$ is an IFC solution of (3.1)–(3.3).

(T2) $\tilde{\mathcal{T}}_{\mathrm{path}}(S^{\mathbb{N}})$ is $\mathbf{P_s}$-trivial.

(T3) $\tilde{\mathcal{T}}_{\mathrm{path}}^{[1]}(S^{\mathbb{N}};\mathbf{P_s})$ is independent of IFC solutions $\bar{P}_{\mathbf{s}}$.

The following is the main result of this section.

Proposition 4.1. The following holds.
(1) Assume **(T1)** and **(T2)**. Then **(P3)** and **(P4)** hold.
(2) Assume **(T1)**–**(T3)**. Then **(P5)** holds.

Let F^∞ be the map in (3.10). The map F^∞ is well defined by Lemma 3.1 and **(T1)**. Note that

$$F^\infty(\mathbf{s},\mathbf{X},\mathbf{B}) = \mathbf{X} \quad \text{for } \bar{P}_{\mathbf{s}}\text{-a.s. } (\mathbf{X},\mathbf{B}). \tag{4.4}$$

Then from this we define the $C_T(S^{\mathbb{N}})$-valued map $F^\infty_{\mathbf{s},\mathbf{B}}$ as

$$F^\infty_{\mathbf{s},\mathbf{B}}(\cdot) = F^\infty(\mathbf{s},\cdot,\mathbf{B}). \tag{4.5}$$

The map $F^\infty_{\mathbf{s},\mathbf{B}}$ is defined for $\bar{P}_{\mathbf{s},\mathbf{B}}$-a.s., where $\bar{P}_{\mathbf{s},\mathbf{B}}$ is defined by (3.20). Similarly as $F^\infty_{\mathbf{s},\mathbf{B}}$, we set for $m\in\mathbb{N}$

$$F^m_{\mathbf{s},\mathbf{B}}(\cdot) = F^m(\mathbf{s},\cdot,\mathbf{B}).$$

The map $F^m_{\mathbf{s},\mathbf{B}}$ is defined for all $\mathbf{X}\in W_{T,\mathrm{sol}}$ unlike $F^\infty_{\mathbf{s},\mathbf{B}}$. $Wfix$ の定義を確認 []

Lemma 4.2. Let $W_{T,\mathrm{fix}} = \{\mathbf{X}\in W_{T,\mathrm{sol}}\,;\ (4.6) \text{ holds for } P^\infty_{\mathrm{Br}}\text{-a.s.}\,\mathbf{B}\}$, where

$$F^\infty_{\mathbf{s},\mathbf{B}}(\mathbf{X}) = \mathbf{X} \text{ for } \bar{P}_{\mathbf{s},\mathbf{B}}\text{-a.s.}. \tag{4.6}$$

Then

$$\bar{P}_{\mathbf{s},\mathbf{B}}(W_{T,\mathrm{fix}}) = 1 \quad \text{for } P^\infty_{\mathrm{Br}}\text{-a.s.}\,\mathbf{B}. \tag{4.7}$$

Proof. We deduce (4.7) from (4.4) immediately. □

We next prepare a general fact on determination class. For a measurable space $(U,\mathcal{U})$ we call a collection $\mathcal{V}$ of elements of $\mathcal{U}$ a determination class of $(U,\mathcal{U})$ if any two probability measures P and Q on $(U,\mathcal{U})$ are equal if and only if $P(A)=Q(A)$ for all $A\in\mathcal{V}$. We say a determination class $\mathcal{V}$ is countable if its cardinality is countable.

Lemma 4.3. Let $(U,\mathcal{U})$ be a measurable space with a countable determination class $\mathcal{V}=\{V_n\}_{n\in\mathbb{N}}$. Let m be a probability measure on $(U,\mathcal{U})$. Suppose that $m(V_n)\in\{0,1\}$ for all $V_n\in\mathcal{V}$. Then $m(A)\in\{0,1\}$ for all $A\in\mathcal{U}$.

Proof. Let $N(1) = \{n \in \mathbb{N}; m(V_n) = 1\}$. If $N(1) = \emptyset$, then m is the zero measure. If $N(1) \neq \emptyset$, then we take

$$V = \Big(\bigcap_{n \in N(1)} V_n \Big) \bigcap \Big(\bigcap_{n \notin N(1)} V_n^c \Big).$$

Clearly, we obtain $m(V) = 1$.

Let $A \in \mathcal{U}$. Suppose that $V \cap A \notin \{\emptyset, V\}$. Then we can not determine the value of $m(V \cap A)$ by the value of $m(V_n)$ $(n \in \mathbb{N})$. This yields contradiction. Hence $V \cap A \in \{\emptyset, V\}$. If $V \cap A = \emptyset$, then $m(A) = 0$. If $V \cap A = V$, then $m(A) \geq m(V) = 1$. Hence we complete the proof. □

Since $S^{\mathbb{N}}$ is a Polish space with the product topology, $C_T(S^{\mathbb{N}})$ becomes a Polish space. Hence there exists a countable determination class $\mathcal{V}$ of its path space of $(C_T(S^{\mathbb{N}}), \mathcal{B}(C_T(S^{\mathbb{N}})))$. We take such a class $\mathcal{V}$ as follows: Let $\mathbf{S}_1$ be a countable dense subset of $S^{\mathbb{N}}$, and

$$\mathcal{U} = \mathcal{A}[\{U_r(\mathbf{s});\, 0 < r \in \mathbb{Q},\, \mathbf{s} \in \mathbf{S}_1\}].$$

Here $\mathcal{A}[\cdot]$ denotes the algebra generated by $\cdot$, and $U_r(\mathbf{s})$ is an open ball in $S^{\mathbb{N}}$ with center $\mathbf{s}$ and radius r. Here we take a suitable metric defining the same topology of the Polish space $S^{\mathbb{N}}$. We note that $\mathcal{U}$ is countable because the subset $\{U_r(\mathbf{s});\, 0 < r \in \mathbb{Q},\, \mathbf{s} \in \mathbf{S}_1\}$ is countable. Let

$$\mathcal{V} = \bigcup_{l=1}^{\infty} \{(\mathbf{X}_{\mathbf{t}})^{-1}(\mathbf{A})\,;\, \mathbf{A} \in \mathcal{U}^l, \mathbf{t} \in (\mathbb{Q} \cap [0,T])^l\}. \tag{4.8}$$

Then $\mathcal{V}$ becomes a countable determination class of $(C_T(S^{\mathbb{N}}), \mathcal{B}(C_T(S^{\mathbb{N}})))$.

Lemma 4.4. Assume **(T1)**. Then, for each $\mathbf{V} \in \mathcal{V}$,

$$(F^{\infty}_{\mathbf{s},\mathbf{B}})^{-1}(\mathbf{V}) \bigcap W_{T,\mathrm{fix}} \in \tilde{\mathcal{T}}_{\mathrm{path}}(S^{\mathbb{N}}) \bigcap W_{T,\mathrm{fix}} \quad \text{for } P^{\infty}_{\mathrm{Br}}\text{-a.s. } \mathbf{B}. \tag{4.9}$$

Proof. By Lemma 3.1 (3), the function $F^{\infty}_{\mathbf{s},\mathbf{B}}$ is $\mathcal{T}_{\mathrm{path}}(S^{\mathbb{N}})$-measurable for P^{∞}_{Br}-a.s. $\mathbf{B}$. Hence

$$(F^{\infty}_{\mathbf{s},\mathbf{B}})^{-1}(\mathbf{V}) \in \mathcal{T}_{\mathrm{path}}(S^{\mathbb{N}}) \quad \text{for } P^{\infty}_{\mathrm{Br}}\text{-a.s. } \mathbf{B}. \tag{4.10}$$

Let $\mathbf{X} \in (F^{\infty}_{\mathbf{s},\mathbf{B}})^{-1}(\mathbf{V}) \bigcap W_{T,\mathrm{fix}}$. Then, since $\mathbf{X} \in W_{T,\mathrm{fix}}$, we deduce that

$$F^{\infty}_{\mathbf{s},\mathbf{B}}(\mathbf{X}) = \mathbf{X} \tag{4.11}$$

Hence we see that

$$(F^{\infty}_{\mathbf{s},\mathbf{B}})^{-1}(\mathbf{V}) \bigcap W_{T,\mathrm{fix}} = \mathbf{V} \bigcap W_{T,\mathrm{fix}} \quad \text{for } P^{\infty}_{\mathrm{Br}}\text{-a.s. } \mathbf{B}. \tag{4.12}$$

Since $\mathbf{V} \in \mathcal{V}$, there exists a $\mathbf{t}$ such that $\mathbf{V} \in \sigma[\mathbf{X}_{\mathbf{t}}]$. Then

$$\mathbf{V} \cap W_{T,\mathrm{fix}} \in \sigma[\mathbf{X}_{\mathbf{t}}] \cap W_{T,\mathrm{fix}}. \tag{4.13}$$

Combining (4.10), (4.12), and (4.13) yields (4.9). □

We now recall a decomposition of $\mathbf{P_s}$. From (3.20), we deduce that

$$\mathbf{P_s}(\mathbf{A}) = \int_{C_T^{\mathbf{0}}(S^{\mathbb{N}})} \bar{P}_{\mathbf{s},\mathbf{B}}(\mathbf{A}) P_{\mathrm{Br}}^{\infty}(d\mathbf{B}). \tag{4.14}$$

Lemma 4.5. Assume **(T2)**. Then, for each $\mathbf{A} \in \tilde{\mathcal{T}}_{\mathrm{path}}(S^{\mathbb{N}})$,

$$\bar{P}_{\mathbf{s},\mathbf{B}}(\mathbf{A}) \in \{0,1\} \quad \text{for } P_{\mathrm{Br}}^{\infty}\text{-a.s. } \mathbf{B}. \tag{4.15}$$

Proof. Suppose that (4.15) is false. Let $B_{\mathbf{A}} = \{\mathbf{B}\,;\, 0 < \bar{P}_{\mathbf{s},\mathbf{B}}(\mathbf{A}) < 1\}$. Then there exists an $\mathbf{A} \in \tilde{\mathcal{T}}_{\mathrm{path}}(S^{\mathbb{N}})$ such that $P_{\mathrm{Br}}^{\infty}(B_{\mathbf{A}}) > 0$. Hence we deduce from the definition of $B_{\mathbf{A}}$ and $P_{\mathrm{Br}}^{\infty}(B_{\mathbf{A}}) > 0$ that

$$0 < \int_{C_T^{\mathbf{0}}(S^{\mathbb{N}})} \bar{P}_{\mathbf{s},\mathbf{B}}(\mathbf{A}) P_{\mathrm{Br}}^{\infty}(d\mathbf{B}) < 1. \tag{4.16}$$

We deduce from (4.14) and (4.16) that $0 < \mathbf{P_s}(\mathbf{A}) < 1$, which contradicts **(T2)**. ☐

Proposition 4.6. Assume **(T1)** and **(T2)**. Then **(P3)** holds.

Proof. From (4.4) we deduce that, for P_{Br}^{∞}-a.s. $\mathbf{B}$,

$$\bar{P}_{\mathbf{s},\mathbf{B}} \circ (F_{\mathbf{s},\mathbf{B}}^{\infty})^{-1} = \bar{P}_{\mathbf{s},\mathbf{B}}. \tag{4.17}$$

We deduce from Lemma 4.4, Lemma 4.5, and $F_{\mathbf{s},\mathbf{B}}^{\infty}(W_{T,\mathrm{fix}}) = 1$, that, for all $\mathbf{V} \in \mathcal{V}$,

$$\bar{P}_{\mathbf{s},\mathbf{B}} \circ (F_{\mathbf{s},\mathbf{B}}^{\infty})^{-1}(\mathbf{V}) \in \{0,1\} \quad \text{for } P_{\mathrm{Br}}^{\infty}\text{-a.s. } \mathbf{B}.$$

Since $\mathcal{V}$ is countable, we deduce that, for P_{Br}^{∞}-a.s. $\mathbf{B}$,

$$\bar{P}_{\mathbf{s},\mathbf{B}} \circ (F_{\mathbf{s},\mathbf{B}}^{\infty})^{-1}(\mathbf{V}) \in \{0,1\} \quad \text{for all } \mathbf{V} \in \mathcal{V}. \tag{4.18}$$

Since $\mathcal{V}$ is a countable determination class, we obtain from (4.18) and Lemma 4.3 that

$$\bar{P}_{\mathbf{s},\mathbf{B}} \circ (F_{\mathbf{s},\mathbf{B}}^{\infty})^{-1}(\mathbf{A}) \in \{0,1\} \quad \text{for all } \mathbf{A} \in \mathcal{B}(C_T(S^{\mathbb{N}})). \tag{4.19}$$

Hence we deduce that $\bar{P}_{\mathbf{s},\mathbf{B}} \circ (F_{\mathbf{s},\mathbf{B}}^{\infty})^{-1} = \delta_{\mathbf{X}}$ for some $\mathbf{X} = \mathbf{X}(\mathbf{s},\mathbf{B}) \in C_T(S^{\mathbb{N}})$. In particular, $\mathbf{X}$ is a function of $(\mathbf{s},\mathbf{B})$. This combined with (4.17) implies that $\bar{P}_{\mathbf{s},\mathbf{B}} = \delta_{\mathbf{X}}$. From this we immediately obtain **(P3)**. ☐

The following two lemmas are corollaries of Proposition 4.6.

Lemma 4.7. Assume **(T1)** and **(T2)**. Then **(P4)** holds.

Proof. For $\mathbf{A} \in \mathcal{T}_{\mathrm{path}}(S^{\mathbb{N}})$ let $\mathcal{W} = \{\mathbf{B}\,;\, \bar{P}_{\mathbf{s},\mathbf{B}}(\mathbf{A}) = 1\}$. Recall that $\bar{P}_{\mathbf{s},\mathbf{B}}(\mathbf{A}) \in \{0,1\}$ from Proposition 4.6. Then we deduce from (4.14) that

$$\mathbf{P_s}(\mathbf{A}) = \int_{\mathcal{W}} \bar{P}_{\mathbf{s},\mathbf{B}}(\mathbf{A}) P_{\mathrm{Br}}^{\infty}(d\mathbf{B}) = P_{\mathrm{Br}}^{\infty}(\mathcal{W}). \tag{4.20}$$

Hence $\mathbf{P_s}(\mathbf{A}) = 1$ if and only if $\bar{P}_{\mathbf{s},\mathbf{B}}(\mathbf{A}) = 1$ for P_{Br}^{∞}-a.s. $\mathbf{B}$, from which we deduce **(P4)**. ☐

[] 証明まずい。まだ書いていない。(T3) を使わないといけない。

Lemma 4.8. Assume **(T1)**–**(T3)**. Then **(P5)** holds.

Proof. From the proof of Lemma 4.7 we see that $\mathbf{A} \in \mathcal{T}_{\mathrm{path}}^{[1]}(S^{\mathbb{N}}; \mathbf{P_s})$ if and only if $\bar{P}_{\mathbf{s},\mathbf{B}}(\mathbf{A}) = 1$ for P_{Br}^{∞}-a.s. $\mathbf{B}$. From this we deduce **(P5)**. ☐

Proof of Proposition 4.1. Proposition 4.1 follows from Proposition 4.6, Lemma 4.7 and Lemma 4.8, immediately. ☐

5 Cylindrical tail triviality of the unlabeled path space $C_T(S^{\mathbb{N}})$

The purpose of this section is to deduce the tail triviality of the labeled path spaces $C_T(S^{\mathbb{N}})$ from that of the configuration space S through that of unlabeled the path space $C_T(\mathsf{S})$. The proof consists of two steps: (i) the derivation of the cylindrical tail triviality of $C_T(\mathsf{S})$ from the tail triviality of S, and (ii) the derivation of the tail triviality of $C_T(S^{\mathbb{N}})$ from its cylindrical tail triviality. We will carry out (i) in Subsection 5.1, and (ii) in Subsection 5.2.

We note that the space S is a *tiny* infinite-dimensional space compare with $S^{\mathbb{N}}$. Hence we usually have a nice probability measure μ and μ-invariant stochastic dynamics on it. This fact is a key to the proof of this derivation.

5.1 From the tail triviality of S to the cylindrical tail triviality of $C_T(\mathsf{S})$.

Let $\mathcal{T}(\mathsf{S})$ be as in (2.33). We will lift the μ-triviality of $\mathcal{T}(\mathsf{S})$ to the cylindrical tail triviality of the unlabeled path space $C_T(\mathsf{S})$. For this we equip a probability measures P_μ on $C_T(\mathsf{S})$. We call P_μ lift dynamics, which lift the measure μ to that of the unlabeled path space $C_T(\mathsf{S})$. Later in Subsection 5.2, we will further lift them to the measures on the labeled path space $C_T(S^{\mathbb{N}})$ through the path label map $\mathfrak{l}_{\mathrm{path}}$ in (2.27).

We write $\mathsf{X} = \{\mathsf{X}_t\}_{0 \le t \le T} \in C_T(\mathsf{S})$ and set $\mathsf{P}_\mu^{\mathsf{X}_t} = \mathsf{P}_\mu \circ \mathsf{X}_t^{-1}$. Then we assume the following.

(Q1) There exists a probability measures P_μ on $C_T(\mathsf{S})$ satisfying

$$\mathsf{P}_\mu^{\mathsf{X}_t} \prec \mu \text{ for all } 0 \le t \le T,. \tag{5.1}$$

Remark 5.1. (1) The assumption **(Q1)** is quite mild. Indeed, if $\{\mathsf{P_s}\}$ is a Markov process with state space $\mathsf{S}_0 \subset \mathsf{S}$ such that $\mu(\mathsf{S}_0) = 1$ and that $\{\mathsf{P_s}\}$ is μ-symmetric, then $\mathsf{P}_\mu = \int \mathsf{P_s} d\mu$ satisfies **(Q1)**.

(2) Let $\mathsf{P_s}$ denote the regular conditional probability: $\mathsf{P_s}(\,\cdot\,) = \mathsf{P}_\mu(\,\cdot\,|\mathsf{X}_0 = \mathsf{s})$. We remark that we do not assume $\{\mathsf{P_s}\}$ is μ-reversible diffusion in general. This point may be useful when one consider

non-Markov and non-symmetric type ISDEs.
(3) We will later introduce a Borel subset H of S, and consider H-valued path measures in **(Q1)** instead of S-valued one. By posing suitable assumptions on H, we will convert the unlabeled paths to labeled paths. The property $\mathsf{H} \subset \mathsf{S}_{\text{s.i.}} \subset \mathsf{S}$ will be used there.

For a probability P defined on $\mathcal{T}(\mathsf{S})$ we set $\mathcal{T}^{[1]}(\mathsf{S}, P) = \{\mathcal{X} \in \mathcal{T}(\mathsf{S})\,;\, P(\mathcal{X}) = 1\}$.

Lemma 5.1. Assume **(A9)** and **(Q1)**. Then for each t, the following holds.
(1) $\mathcal{T}(\mathsf{S})$ is $\mathsf{P}_{\mu}^{\mathsf{X}_t}$-trivial.
(2) $\mathcal{T}^{[1]}(\mathsf{S}, \mu) = \mathcal{T}^{[1]}(\mathsf{S}, \mathsf{P}_{\mu}^{\mathsf{X}_t})$.

Remark 5.2. From Lemma 5.1 (2) we see that $\mathcal{T}^{[1]}(\mathsf{S}, \mathsf{P}_{\mu}^{\mathsf{X}_t})$ is independent of the particular choice of P_{μ} in **(Q1)**.

Proof. From (5.1) and **(A9)** we obtain (1). We deduce from (5.1) and (1) that $\mathcal{X} \in \mathcal{T}^{[1]}(\mathsf{S}, \mathsf{P}_{\mu}^{\mathsf{X}_t})$ if and only if $\mathcal{X} \in \mathcal{T}^{[1]}(\mathsf{S}, \mu)$. We have thus obtained (2). □

Lemma 5.2. Assume **(A9)** and **(Q1)**. Then for each t, the following holds.
(1) $\mathcal{T}(\mathsf{S})$ is $\mathsf{P}_{\mathsf{s}}^{\mathsf{X}_t}$-trivial for μ-a.s. s.
(2) $\mathcal{T}^{[1]}(\mathsf{S}, \mu) = \mathcal{T}^{[1]}(\mathsf{S}, \mathsf{P}_{\mathsf{s}}^{\mathsf{X}_t})$ μ-a.s. s.

Proof. The proof of (1) is similar to that of Lemma 4.5. Hence we omit it.
From Lemma 5.1, we deduce (2) from $\mathcal{T}^{[1]}(\mathsf{S}, \mathsf{P}_{\mu}^{\mathsf{X}_t}) = \mathcal{T}^{[1]}(\mathsf{S}, \mathsf{P}_{\mathsf{s}}^{\mathsf{X}_t})$ μ-a.s. s. The proof of this is similar to that of Lemma 4.8. □

We write $\mathsf{X} = \{\mathsf{X}_t\}_{0 \le t \le T} \in C_T(\mathsf{S})$. Let

$$\tilde{\mathcal{T}}_{\text{path}}(\mathsf{S}) = \bigvee_{\mathbf{t} \in \mathbf{T}} \bigcap_{r=1}^{\infty} \sigma[\, \pi_r^c(\mathsf{X}_{\mathbf{t}})\,]. \tag{5.2}$$

Here $\pi_r^c(\mathsf{X}_{\mathbf{t}}) = (\pi_r^c(\mathsf{X}_{t_1}), \dots, \pi_r^c(\mathsf{X}_{t_n})) \in \mathsf{S}^n$, where $\mathbf{t} = (t_1, \dots, t_n)$. Hence $\tilde{\mathcal{T}}_{\text{path}}(\mathsf{S})$ is a sub σ-field of $\mathcal{B}(C_T(\mathsf{S}))$, and is called the cylindrical tail σ-field of $C_T(\mathsf{S})$. We set

$$\tilde{\mathcal{T}}_{\text{path}}^{[1]}(\mathsf{S}; P) = \{\mathcal{X} \in \tilde{\mathcal{T}}_{\text{path}}(\mathsf{S})\,;\, P(\mathcal{X}) = 1\}. \tag{5.3}$$

Lemma 5.3. Assume **(A9)** and **(Q1)**. Let μ be fixed. Then the following holds.
(1) $\tilde{\mathcal{T}}_{\text{path}}(\mathsf{S})$ is P_{μ}-trivial.
(2) $\tilde{\mathcal{T}}_{\text{path}}^{[1]}(\mathsf{S}; \mathsf{P}_{\mu})$ is independent of the lift dynamics P_{μ} in **(Q1)**.

Proof. For $N \in \mathbb{N}$, $0 \le t_1 < t_2 < \cdots < t_N \le T$, and $\mathsf{A}_i \in \mathcal{T}(\mathsf{S})$, let

$$\mathcal{X} = \{\mathsf{X} \in C_T(\mathsf{S})\,;\, \mathsf{X}_{t_i} \in \mathsf{A}_i \ (i = 1, \dots, N)\}. \tag{5.4}$$

Let $\mathit{CTail}\,(C_T(\mathsf{S}))$ denote the subset of $\tilde{\mathcal{T}}_{\mathrm{path}}(\mathsf{S})$ of the form (5.4). Namely, $\mathit{CTail}\,(C_T(\mathsf{S}))$ consists of the cylindrical sets of $\tilde{\mathcal{T}}_{\mathrm{path}}(\mathsf{S})$. Let $\mathcal{X} \in \mathit{CTail}\,(C_T(\mathsf{S}))$ be as in (5.4). Then from Lemma 5.1, we deduce that

$$\mathsf{P}_{\mathsf{s}}(\mathsf{X}_{t_i} \in \mathsf{A}_i) \in \{0,1\} \text{ for all } i$$

and that

$$1_{\mathsf{A}_i}(\mathsf{s}) = \mathsf{P}_{\mathsf{s}}(\mathsf{X}_{t_i} \in \mathsf{A}_i) \quad \mu\text{-a.s.}\,\mathsf{s}.$$

From this and the Markov property of X we deduce that (1) and (2) hold if we replace $\tilde{\mathcal{T}}_{\mathrm{path}}(\mathsf{S})$ by $\mathit{CTail}\,(C_T(\mathsf{S}))$. Then by the monotone class theorem, we refine this from $\mathit{CTail}\,(C_T(\mathsf{S}))$ to $\tilde{\mathcal{T}}_{\mathrm{path}}(\mathsf{S})$. This completes the proof. □

5.2 From the cylindrical tail triviality of $C_T(\mathsf{S})$ to that of $C_T(S^{\mathbb{N}})$.

To construct labeled dynamics from unlabeled one, we introduce the condition **(Q2)** below. This condition corresponds to **(A4)** in Section 2.

(Q2) $\mathsf{P}_{\mu}(C_T(\mathsf{S}_{\mathrm{s.i.}})) = 1$.

Lemma 5.4. Assume **(A9)**, **(Q1)**, and **(Q2)**. We set

$$\mathcal{T}(\mathsf{S}_{\mathrm{s.i.}}) = \{\mathsf{A} \cap \mathsf{S}_{\mathrm{s.i.}}\,;\, \mathsf{A} \in \mathcal{T}(\mathsf{S})\}\ ,\ \ \tilde{\mathcal{T}}_{\mathrm{path}}(\mathsf{S}_{\mathrm{s.i.}}) = \{\mathsf{W} \cap C_T(\mathsf{S}_{\mathrm{s.i.}}); \mathsf{W} \in \tilde{\mathcal{T}}_{\mathrm{path}}(\mathsf{S})\}.$$

Then $\mu(\mathsf{S}_{\mathrm{s.i.}}) = 1$, $\mathcal{T}(\mathsf{S}_{\mathrm{s.i.}})$ is μ-trivial, and

$$\tilde{\mathcal{T}}_{\mathrm{path}}(\mathsf{S}) = \tilde{\mathcal{T}}_{\mathrm{path}}(\mathsf{S}_{\mathrm{s.i.}}) \quad \text{for } \mathsf{P}_{\mu}\text{-a.s.}. \tag{5.5}$$

Proof. From **(Q2)** we deduce that $\mathsf{P}_{\mu}^{\mathsf{X}_t}(\mathsf{S}_{\mathrm{s.i.}}) = 1$ for any t. This together with **(Q1)** yields $\mu(\mathsf{S}_{\mathrm{s.i.}}) = 1$. From **(A9)** and $\mu(\mathsf{S}_{\mathrm{s.i.}}) = 1$, we deduce that $\mathcal{T}(\mathsf{S}_{\mathrm{s.i.}})$ is μ-trivial. The claim (5.5) is obvious from **(Q2)**. □

Let $\mathfrak{l}(\mathsf{s}) = (\mathfrak{l}_n(\mathsf{s}))_{n\in\mathbb{N}}$ be a measurable label and let $\mathfrak{l}_{\mathrm{path}}$ be the associated label path map $\mathfrak{l}_{\mathrm{path}} : C_T(\mathsf{S}_{\mathrm{s.i.}}) \to C_T(S^{\mathbb{N}})$ defined by (2.27). We write $\mathsf{X} = \sum_{n\in\mathbb{N}} \delta_{X^n}$ and $\mathbf{X} = (X^n)_{n\in\mathbb{N}}$, where $X^n = \mathfrak{l}_{\mathrm{path}}^n(\mathsf{X}) \in C([0,T]; S)$. By definition $\mathsf{X}_t = \sum_{n=1}^{\infty} \delta_{X_t^n}$ and $\mathbf{X}_t = (X_t^n)_{n\in\mathbb{N}}$ for all t.

We set $C_T(S_r^c) = C([0,T]; S_r^c)$ and define $\mathsf{m}_{r,T} : C_T(\mathsf{S}_{\mathrm{s.i.}}) \to \mathbb{N} \cup \{\infty\}$ as

$$\mathsf{m}_{r,T}(\mathsf{X}) = \inf\{m \in \mathbb{N}\,;\, X^n \in C_T(S_r^c) \text{ for all } m < n \in \mathbb{N}\}. \tag{5.6}$$

Here we regard X^n as a map from $\mathsf{S}_{\mathrm{s.i.}}$ to $C_T(S)$ by the correspondence $\mathsf{X} = \sum_{i=1}^{\infty} \delta_{X^i} \mapsto X^n$. By construction, this map is the composition of the path coordinate map $\mathbf{X} = (X^i)_{i\in\mathbb{N}} \mapsto X^n$ with $\mathfrak{l}_{\mathrm{path}}$, and depends on the label $\mathfrak{l}$ since $\mathfrak{l}_{\mathrm{path}}$ does. We introduce the condition:

(Q3) The probability measure P_{μ} and the label $\mathfrak{l}$ satisfy the following. For each $T \in \mathbb{N}$

$$\mathsf{P}_{\mu}(\bigcap_{r=1}^{\infty}\{\mathsf{m}_{r,T}(\mathsf{X}) < \infty\}) = 1. \tag{5.7}$$

The assumption **(Q3)** is a key to the lift from the unlabeled path space to the labeled path space. We use **(Q3)** in Lemma 5.5 to control the fluctuation of trajectories of the *labeled* path $\mathbf{X}$.

Lemma 5.5. Assume **(A9)**, and **(Q1)**–**(Q3)** and let $\tilde{\mathcal{T}}_{\mathrm{path}}(S^{\mathbb{N}})$ be as in (4.1). Then

$$\mathfrak{l}_{\mathrm{path}}^{-1}(\tilde{\mathcal{T}}_{\mathrm{path}}(S^{\mathbb{N}})) \subset \tilde{\mathcal{T}}_{\mathrm{path}}(\mathsf{S}_{\mathrm{s.i.}}) \quad \text{for } \mathsf{P}_{\mu}\text{-a.s.}. \tag{5.8}$$

Proof. Recall that $\tilde{\mathcal{T}}_{\mathrm{path}}(S^{\mathbb{N}}) = \bigvee_{\mathbf{t}\in\mathbf{T}} \bigcap_{n=1}^{\infty} \sigma[\mathbf{X}_{\mathbf{t}}^{n*}]$. Let $\mathbf{A} \in \bigcup_{\mathbf{t}\in\mathbf{T}} \bigcap_{n=1}^{\infty} \sigma[\mathbf{X}_{\mathbf{t}}^{n*}]$ be an arbitrary element. Then for some $\mathbf{t} = (t_1, \ldots, t_k) \in \mathbf{T}$,

$$\mathbf{A} \in \bigcap_{n=1}^{\infty} \sigma[\mathbf{X}_{\mathbf{t}}^{n*}].$$

We set $\pi_{S_r^c}(\mathsf{X}_{\mathbf{t}}) = (\pi_{S_r^c}(\mathsf{X}_{t_1}), \ldots, \pi_{S_r^c}(\mathsf{X}_{t_k}))$. From these, we deduce that

$$\mathfrak{l}_{\mathrm{path}}^{-1}(\mathbf{A}) \bigcap \{\mathsf{m}_{r,T}(\mathsf{X}) < \infty\} \in \sigma[\pi_{S_r^c}(\mathsf{X}_{\mathbf{t}})] \bigcap \{\mathsf{m}_{r,T}(\mathsf{X}) < \infty\} \quad \text{for all } r \in \mathbb{N}. \tag{5.9}$$

Combining **(Q3)** and (5.9), we obtain that for P_{μ}-a.s.

$$\begin{aligned} \mathfrak{l}_{\mathrm{path}}^{-1}(\mathbf{A}) &= \bigcap_{r=1}^{\infty} \{\mathfrak{l}_{\mathrm{path}}^{-1}(\mathbf{A}) \bigcap \{\mathsf{m}_{r,T}(\mathsf{X}) < \infty\}\} \\ &\in \bigcap_{r=1}^{\infty} \{\sigma[\pi_{S_r^c}(\mathsf{X}_{\mathbf{t}})] \bigcap \{\mathsf{m}_{r,T}(\mathsf{X}) < \infty\}\} \\ &= \bigcap_{r=1}^{\infty} \sigma[\pi_{S_r^c}(\mathsf{X}_{\mathbf{t}})] \subset \tilde{\mathcal{T}}_{\mathrm{path}}(\mathsf{S}). \end{aligned} \tag{5.10}$$

Hence we deduce from (5.10) and Lemma 5.4 that $\mathfrak{l}_{\mathrm{path}}^{-1}(\mathbf{A}) \in \tilde{\mathcal{T}}_{\mathrm{path}}(\mathsf{S}_{\mathrm{s.i.}})$ for P_{μ}-a.s.. Since (5.10) holds for arbitrary $\mathbf{A} \in \bigcup_{\mathbf{t}\in\mathbf{T}} \bigcap_{n=1}^{\infty} \sigma[\mathbf{X}_{\mathbf{t}}^{n*}]$, we obtain (5.8) by the monotone class theorem. □

Similarly as (3.14), we set

$$\tilde{\mathcal{T}}_{\mathrm{path}}^{[1]}(S^{\mathbb{N}}; P) = \{\mathbf{A} \in \tilde{\mathcal{T}}_{\mathrm{path}}(S^{\mathbb{N}}); P(\mathbf{A}) = 1\}. \tag{5.11}$$

We now come to the main result of this section.

Proposition 5.6. Assume **(A9)**, and **(Q1)**–**(Q3)**. Then the following holds.

(1) $\tilde{\mathcal{T}}_{\mathrm{path}}(S^{\mathbb{N}})$ is $\mathbf{P}_{\mu^{\mathfrak{l}}}$-trivial.

(2) $\tilde{\mathcal{T}}_{\mathrm{path}}^{[1]}(S^{\mathbb{N}}; \mathbf{P}_{\mu^{\mathfrak{l}}})$ is independent of the lift dynamics P_{μ} in **(Q1)**.

Proof. Proposition 5.6 is immediate from Lemma 5.3 and Lemma 5.5. □

We next refine Proposition 5.6 to an almost sure statement. Let

$$\mathbf{P}_{\mathbf{s}} = \mathbf{P}_{\mu^{\mathfrak{l}}}(\cdot|\mathbf{X}_0 = \mathbf{s}). \tag{5.12}$$

We assume:

(Q4) There exists a Borel set $\mathbf{S}_0$ such that $\mu^{\mathfrak{l}}(\mathbf{S}_0) = 1$ and that $\tilde{\mathcal{T}}_{\mathrm{path}}(S^{\mathbb{N}})$ is countably determined with $\mathcal{V}_1$ under $\{\mathbf{P}_{\mathbf{s}}\}_{\mathbf{s}\in\mathbf{S}_0}$.

Proposition 5.7. Assume **(A9)** and **(Q1)**–**(Q4)**. Then there exists a Borel set $\mathbf{S}_1$ such that $\mu^{\mathfrak{l}}(\mathbf{S}_1) = 1$ and that $\mathbf{S}_1 \subset \mathbf{S}_0$ satisfying the following.
(1) $\tilde{\mathcal{T}}_{\mathrm{path}}(\mathbf{S}_1)$ is $\mathbf{P}_{\mathbf{s}}$-trivial for all $\mathbf{s} \in \mathbf{S}_1$.
(2) $\tilde{\mathcal{T}}_{\mathrm{path}}^{[1]}(S^{\mathbb{N}};\mathbf{P}_{\mathbf{s}})$ is independent of the lift dynamics P_{μ} in **(Q1)** for all $\mathbf{s} \in \mathbf{S}_1$.

Proof. Let $\mathbf{A} \in \mathcal{V}_1$ be an arbitrary element. Then, from $\mathbf{A} \in \tilde{\mathcal{T}}_{\mathrm{path}}(S^{\mathbb{N}})$, we see that $\mathbf{P}_{\mu^{\mathfrak{l}}}(\mathbf{A}) \in \{0,1\}$ by Proposition 5.6. This together with (5.12) yields

$$\int_{\mathbf{S}} \mathbf{P}_{\mathbf{s}}(\mathbf{A})\mu^{\mathfrak{l}}(d\mathbf{s}) = \mathbf{P}_{\mu^{\mathfrak{l}}}(\mathbf{A}) \in \{0,1\}. \tag{5.13}$$

From this we easily deduce that for all $\mathbf{A} \in \mathcal{V}_1$

$$\mathbf{P}_{\mathbf{s}}(\mathbf{A}) \in \{0,1\} \text{ for } \mu^{\mathfrak{l}}\text{-a.s.}\,\mathbf{s}. \tag{5.14}$$

Hence, by interchanging the role of $\mathbf{A}$ and $\mathbf{s}$, we see that, for $\mu^{\mathfrak{l}}$-a.s. $\mathbf{s}$,

$$\mathbf{P}_{\mathbf{s}}(\mathbf{A}) \in \{0,1\} \text{ for all } \mathbf{A} \in \mathcal{V}_1. \tag{5.15}$$

Since $\mathcal{V}_1$ determines $\tilde{\mathcal{T}}_{\mathrm{path}}(S^{\mathbb{N}})$ under $\{\mathbf{P}_{\mathbf{s}}\}_{\mathbf{s}\in\mathbf{S}_0}$, we see that (5.15) holds for all $\mathbf{A} \in \tilde{\mathcal{T}}_{\mathrm{path}}(S^{\mathbb{N}})$. Similarly, we see that $\tilde{\mathcal{T}}_{\mathrm{path}}^{[1]}(S^{\mathbb{N}};\mathbf{P}_{\mathbf{s}})$ is independent of P_{μ} in **(Q1)** for $\mu^{\mathfrak{l}}$-a.s. $\mathbf{s}$.

With these results we take a $\mu^{\mathfrak{l}}$-version of $\mathbf{S}_0$ to obtain a subset $\mathbf{S}_1$ satisfying (1) and (2). □

Theorem 5.8. Assume **(A9)** and **(Q1)**–**(Q4)**. Let $\mathbf{S}_1$ and $\mathbf{P}_{\mathbf{s}}$ be as in Proposition 5.7. Then **(T2)** and **(T3)** are satisfied for each $\mathbf{s} \in \mathbf{S}_1$.

Proof. Recall that $\mu^{\mathfrak{l}}(\mathbf{S}_1) = 1$, and that $\tilde{\mathcal{T}}_{\mathrm{path}}(\mathbf{S}_1)$ and $\tilde{\mathcal{T}}_{\mathrm{path}}(S^{\mathbb{N}})$ are cylindrical tail σ-fields on path spaces. Then we see that $\tilde{\mathcal{T}}_{\mathrm{path}}(\mathbf{S}_1) = \tilde{\mathcal{T}}_{\mathrm{path}}(S^{\mathbb{N}})$ $\mathbf{P}_{\mathbf{s}}$-a.s. for $\mu^{\mathfrak{l}}$-a.s. $\mathbf{s}$. Hence we deduce **(T2)** from (1) of Proposition 5.7. The assumption **(T3)** follows from (2) of Proposition 5.7 similarly. □

6 Proof of Theorem 2.1

In this section we devote to the proof of Theorem 2.1. Throughout this and the next section section, S will be $\mathbb{R}^d$ or a closed set satisfying the assumption in Section 2, and S is the configuration space over S.

We begin by constructing unlabeled diffusions (X, P). To prove this it is sufficient to check the form $(\mathcal{E}^{a,\mu}, \mathcal{D}_{\circ}^{a,\mu})$ is closable on $L^2(\mathsf{S}, \mu)$, and its closure is a local, quasi-regular Dirichlet form. We refer to [23] for the definition of quasi-regular Dirichlet forms and related notions.

Lemma 6.1. Assume **(A2)** and **(A3)**. Then $(\mathcal{E}^{a,\mu}, \mathcal{D}_{\circ}^{a,\mu})$ is closable on $L^2(\mathsf{S}, \mu)$, and its closure $(\mathcal{E}^{a,\mu}, \mathcal{D}^{a,\mu})$ is a quasi-regular Dirichlet form on $L^2(\mathsf{S}, \mu)$. Moreover, the associated μ-reversible diffusion $(\mathsf{X}, \{\mathsf{P}_{\mathsf{s}}\}_{\mathsf{s}})$ exists.

Proof. The closability follows from [28, 32]. If (2.20) in **(A3)** is satisfied for $p = \infty$, then quasi-regularity follows from [28, Theorem 1]. In the proof of [28, Theorem 1], the assumption $p = \infty$ is used only in the proof of [28, Lemma 2.4]. In fact, this assumption is used in the second line in 126 p in [28] to derive the inequality

$$\|\mathfrak{J}_{r,\varepsilon}\mathfrak{f} - \mathfrak{f}\|_{L^2(\Theta_r^k, \mu)} \le C_r^k \|\mathfrak{J}_{r,\varepsilon}\mathfrak{f} - \mathfrak{f}\|_{L^2(\Theta_r^k, \Lambda)}. \tag{6.1}$$

By using the Hölder inequality instead, we obtain a similar inequality with replacement of $L^2(\Theta_r^k, \Lambda)$ by $L^p(\Theta_r^k, \Lambda)$ in (6.1), which is sufficient for the proof of [28, Lemma 2.4]. The last claim is immediate from the general theory of local, quasi-regular Dirichlet forms [23, Theorem V.2.13, Proposition V.2.15, Theorem V.1.11]. $\square$

We use Theorem 3.4 to prove Theorem 2.1. Hence we will check **(P1)**–**(P5)** in Theorem 3.4. The assumption **(P1)** follows from the following.

Lemma 6.2. Assume **(A1)**–**(A8)**. Then **(P1)** holds with the solution $(\mathbf{X}, \mathbf{B})$ such that the associated unlabeled process is the diffusion $(\mathsf{X}, \{\mathsf{P}_{\mathsf{s}}\}_{\mathsf{s}})$ in Lemma 6.1. Furthermore, **(P2)** holds.

Proof. Let $\mathbf{X} = \mathfrak{l}_{\text{path}}(\mathsf{X})$. Then the first statement follows from in [31, Theorem 26] immediately. The second follows from **(A8)** $\square$

Lemma 6.3. Assume **(A1)**–**(A8)**. Then **(Q1)**–**(Q4)** hold.

Proof. The condition **(Q1)** is obvious because the unlabeled diffusion $\{(\mathsf{X}, \mathsf{P}_{\mathsf{s}})\}_{\mathsf{s} \in \mathfrak{u}(\mathbf{S}_0)}$ is μ-reversible by Lemma 6.1 and Lemma 6.2. **(Q2)** follows from **(A4)**. Let $\mathsf{M}_{r,T}^n$ and $\mathsf{m}_{r,T}$ be as in (2.28) and (5.6), respectively. Then by definition we easily see that

$$\{\mathsf{m}_{r,T}(\mathsf{X}) < \infty\} = \liminf_{n\to\infty} \{\mathsf{M}_{r,T}^n\}^c = \limsup_{n\to\infty} \mathsf{M}_{r,T}^n. \tag{6.2}$$

Hence from **(A6)** we deduce that, for each $r, T \in \mathbb{N}$,

$$\mathsf{P}_{\mu}(\mathsf{m}_{r,T}(\mathsf{X}) < \infty) = 1 - \mathsf{P}_{\mu}(\limsup_{n\to\infty} \mathsf{M}_{r,T}^n) = 1,$$

which yields **(Q3)** immediately. Let $\mathcal{V}$ and $W_{T,\text{fix}}$ be as in (4.8) and Lemma 4.2, respectively. Then $\mathcal{V}_1 = \mathcal{V} \cap W_{T,\text{fix}}$ satisfies **(Q4)**. $\square$

Lemma 6.4. Assume **(A1)**–**(A9)**. Then **(P3)**–**(P5)** hold.

Proof. We use Proposition 4.1 to prove Lemma 6.4. Hence our task is to check all the assumptions **(T1)**–**(T3)** of Proposition 4.1. We recall that **(P1)** and **(P2)** hold from Lemma 6.2. Hence combining **(P1)** and **(P2)** with Lemma 3.1 (2), we obtain **(T1)**.

From Lemma 6.3, we see that **(Q1)**–**(Q4)** hold. By assumption **(A9)** holds. Hence we deduce **(T2)** and **(T3)** from Theorem 5.8. We have thus completed the proof. □

Proof of Theorem 2.1. From Lemma 6.2 and Lemma 6.4 we see that the assumptions **(P1)**–**(P5)** of Theorem 3.4 are fulfilled. From Theorem 3.4 (1) we deduce the existence of a strong solution $\mathbf{X}$ for each $\mathbf{s} \in \mathbf{S}_0$. In fact, the labeled dynamics $\mathbf{X}$ in Lemma 6.2 becomes a strong solution by Theorem 3.4. Since $\mathbf{X} = \mathfrak{l}_{\mathrm{path}}(\mathsf{X})$ with μ-reversible diffusion $(\mathsf{X}, \{\mathsf{P}_{\mathsf{s}}\}_{\mathsf{s}})$ in Lemma 6.1, $\mathbf{X}$ satisfies (2.35).

[] 初期条件を quasi-every にうまく選べば、確率 1 の出発点で、ずっと解ができることをコメント。 □

7 Proof of Theorem 2.2 and a tail decomposition.

In this section we prove Theorem 2.2. We will relax the assumption **(A9)** of the μ-tail triviality in Theorem 2.1, and deduce Theorem 2.2 from Theorem 2.1. For this we use the decomposition of μ with respect to the tail σ-field $\mathcal{T}(\mathsf{S})$. In fact, we will prove that $\mathcal{T}(\mathsf{S})$ is μ-trivial for each components. The next lemma is well known for Gibbs measures, and reveal the fact that quasi-Gibbs measures inherit it.

Lemma 7.1. Assume **(A2)**. Let $\mu_{\mathrm{Tail}}^{\mathsf{a}}$ be a regular conditional probability given by (2.37). Then there exists a version of $\mu_{\mathrm{Tail}}^{\mathsf{a}}$ such that (2.39) holds for μ-a.s. a. In particular, $\mathcal{T}(\mathsf{S})$ is $\mu_{\mathrm{Tail}}^{\mathsf{a}}$-trivial for μ-a.s. a.

Proof. This lemma follows from Georgii [11, (7.22) Proposition]. Indeed, we take $(\Omega, \mathcal{F})$ and other quantities in [11] to be $(\Omega, \mathcal{F}) = (\mathsf{S}, \mathcal{B}(\mathsf{S}))$, $\mathcal{A} = \mathcal{T}(\mathsf{S})$, $\mathcal{P} = \{\mu_{\mathrm{Tail}}^{\mathsf{a}}\,;\, \mathsf{a} \in \mathsf{S}_0\}$, and

$$\mathcal{P}(\Omega, \mathcal{F}) = \{\mu\} \cup \{\mu(\cdot|\pi_{S_r^c})(\mathsf{s})\,;\, r \in \mathbb{N}, \} \cup \mathcal{P}.$$

Let $\mathcal{P}_{\mathcal{A}} = \{\nu \in \mathcal{P}; \nu(A) \in \{0,1\}\}$ as in [11, (7.22) Proposition]. Then by construction

$$\mathcal{P}_{\mathcal{A}} = \{\mu_{\mathrm{Tail}}^{\mathsf{a}}\,;\, \mathsf{a} \in \mathsf{S}_0,\, \mu_{\mathrm{Tail}}^{\mathsf{a}}(A) \in \{0,1\} \text{ for all } A \in \mathcal{T}(\mathsf{S})\}. \tag{7.1}$$

We take a regular conditional probability $\mu_{\mathrm{Tail}}^{\mathsf{a}} = \mu(\cdot|\mathcal{T}(\mathsf{S}))$ as the specification in [11], and also $\mu_{\mathrm{Tail}}^{\mathsf{a}}$ as the $(\mathcal{P}, \mathcal{A})$-kernels in [11]. Then, from in Georgii [11, (7.22) Proposition 130 p.], we deduce that for each $\nu \in \mathcal{P}$ there exists a unique, probability measure w on $(\mathcal{P}_{\mathcal{A}}, \mathcal{G})$ satisfying

$$\nu = \int_{\mathcal{P}_{\mathcal{A}}} \mu^{\mathsf{a}}_{\mathrm{Tail}} w(d\mu^{\mathsf{a}}_{\mathrm{Tail}}). \tag{7.2}$$

Here $\mathcal{G}$ is the σ-field on $\mathcal{P}_{\mathcal{A}}$ generated by the evaluation map $\{\mathfrak{e}_A\}_{A\in\mathcal{B}(\mathsf{S})}$, where $\mathfrak{e}_A$ is given by $\mathfrak{e}_A(\nu) = \nu(A)$.

Since $\mu \in \mathcal{P}$, we can take $\nu = \mu$ in (7.2). Then we conclude Proposition 7.1 from (7.1) and (7.2). $\square$

We next deduce **(A1$_{\mathsf{a}}$)**–**(A8$_{\mathsf{a}}$)** from **(A1)**–**(A8)**. Namely, the conditions **(A1)**–**(A8)** inherit to those for $\mu^{\mathsf{a}}_{\mathrm{Tail}}$. We begin by **(A1)**–**(A4)**.

Lemma 7.2. Assume **(A1)**–**(A4)**, **(A5')**, **(A7)**, and **(A8)**. Then $\mu^{\mathsf{a}}_{\mathrm{Tail}}$ satisfy **(A1$_{\mathsf{a}}$)**–**(A4$_{\mathsf{a}}$)** **(A5'$_{\mathsf{a}}$)**, **(A7$_{\mathsf{a}}$)**, and **(A8$_{\mathsf{a}}$)** for μ-a.s. a.

Proof. **(A1$_{\mathsf{a}}$)** and **(A2$_{\mathsf{a}}$)** are clear from the definition of logarithmic derivatives and quasi-Gibbs measures combined with Fubini's theorem.

(A3$_{\mathsf{a}}$) follows from **(A3)** and the Fubini's theorem. Indeed, we have

$$\int_{\mathsf{S}} [\sum_{m=1}^{\infty} m\, \mu^{\mathsf{a}}_{\mathrm{Tail}}(\mathsf{S}_r^m)]\mu(d\mathsf{a}) = \sum_{m=1}^{\infty} m \int_{\mathsf{S}} \mu^{\mathsf{a}}_{\mathrm{Tail}}(\mathsf{S}_r^m)\mu(d\mathsf{a}) = \sum_{m=1}^{\infty} m\, \mu(\mathsf{S}_r^m) < \infty. \tag{7.3}$$

Hence, for μ-a.s. a, we see that $\mu^{\mathsf{a}}_{\mathrm{Tail}}$ satisfies (2.19). (2.20) for $\mu^{\mathsf{a}}_{\mathrm{Tail}}$ follows from (2.20) for μ combined with Fubini's theorem and the Hölder inequality. Similarly we have **(A5'$_{\mathsf{a}}$)** through Fubini's theorem.

From **(A2$_{\mathsf{a}}$)** we deduce that $(\mathcal{E}^{\mu^{\mathsf{a}}_{\mathrm{Tail}}}, \mathcal{D}_{\circ})$ is closable on $L^2(\mathsf{S}, \mu^{\mathsf{a}}_{\mathrm{Tail}})$. So let $(\mathcal{E}^{\mu^{\mathsf{a}}_{\mathrm{Tail}}}, \mathcal{D}^{\mu^{\mathsf{a}}_{\mathrm{Tail}}})$ denote its closure. Then, it is not difficult to see that

$$\mathcal{D}^{\mu} \subset \mathcal{D}^{\mu^{\mathsf{a}}_{\mathrm{Tail}}}. \tag{7.4}$$

From (7.4) and the definition of the capacity, we easily deduce that

$$\int_{S} \mathrm{Cap}^{\mu^{\mathsf{a}}_{\mathrm{Tail}}}(\mathsf{S}^c_{\mathrm{s.i.}})\mu(d\mathsf{a}) \le \mathrm{Cap}^{\mu}(\mathsf{S}^c_{\mathrm{s.i.}}). \tag{7.5}$$

By **(A4)** we have $\mathrm{Cap}^{\mu}(\mathsf{S}^c_{\mathrm{s.i.}}) = 0$. This combined with (7.5) yields that $\mathrm{Cap}^{\mu^{\mathsf{a}}_{\mathrm{Tail}}}(\mathsf{S}^c_{\mathrm{s.i.}}) = 0$ for μ-a.s. a. Hence we obtain **(A4$_{\mathsf{a}}$)**. Similarly we have **(A7$_{\mathsf{a}}$)** through the capacity inequality (7.5) with the replacement of $\mathsf{S}_{\mathrm{s.i.}}$ by H. **(A8$_{\mathsf{a}}$)** follows from **(A8)** combined with Fubini's theorem. $\square$

Proof of Theorem 2.2. From Lemma 7.2, we see that $\mu^{\mathsf{a}}_{\mathrm{Tail}}$ satisfies **(A1$_{\mathsf{a}}$)**–**(A4$_{\mathsf{a}}$)** **(A5'$_{\mathsf{a}}$)**, **(A7$_{\mathsf{a}}$)**, and **(A8$_{\mathsf{a}}$)** for μ-a.s. a. From Lemma 7.1, we deduce **(A9)**, that is, $\mathcal{T}(\mathsf{S})$ is $\mu^{\mathsf{a}}_{\mathrm{Tail}}$-trivial for μ-a.s. a. Hence we conclude Theorem 2.2 from Theorem 2.1 and (2.38). $\square$

Remark 7.1. We have two diffusions $\{(\mathsf{X},\mathsf{P}_{\mathsf{s}})\}_{\mathsf{s}\in\mathsf{S}_0}$ and $\{(\mathsf{X},\mathsf{P}_{\mathsf{s}}^{\mathsf{a}})\}_{\mathsf{s}\in\mathsf{S}_0^{\mathsf{a}}}$. The former is deduced from **(A2)** and **(A3)**, and the associated Dirichlet form is given by $(\mathcal{E}^{a,\mu},\mathcal{D}^{a,\mu})$ on $L^2(\mathsf{S}_0,\mu)$. The latter is deduced from **(A2$_{\mathsf{a}}$)** and **(A3$_{\mathsf{a}}$)**, the associated Dirichlet form is $(\mathcal{E}^{\mu^{\mathsf{a}}_{\mathrm{Tail}}},\mathcal{D}^{\mu^{\mathsf{a}}_{\mathrm{Tail}}})$ on $L^2(\mathsf{S}_0^{\mathsf{a}},\mu^{\mathsf{a}}_{\mathrm{Tail}})$. When μ has a trivial tail, then these two diffusions are the same. We note that μ has a trivial tail whenever μ is a determinantal random point field ([34]).

8 Sufficient conditions of (A4)–(A8).

The purpose of this section is to give sufficient conditions of the assumptions **(A1)**–**(A8)** of Theorem 2.2. As we noted in Section 2, a sufficient condition of **(A1)** is given in [31], and those of **(A2)** is given in [32, 33]. **(A3)** is usually easy to check by construction. Hence our task is to give sufficient conditions of **(A4)**–**(A8)**.

Let $(\mathcal{E}^{a,\mu},\mathcal{D}^{a,\mu})$ be the Dirichlet form on $L^2(\mathsf{S},\mu)$ introduced in **(A3)**. Let $\mathfrak{l}$ be a label.

We prepare the Dirichlet forms describing the m-labeled processes $(\mathbf{X}^m,\mathsf{X}^{m*})$ $(m\in\mathbb{N})$. Let $\mathsf{X}=\sum_{i\in\mathbb{N}}\delta_{X^i}$ be the associated unlabeled process and let $\mathfrak{l}$ be a label. From **(A4)** and **(A5)** we construct the fully labeled process $\mathbf{X}=(X^i)_{i\in\mathbb{N}}$ with $\mathbf{X}_0=\mathfrak{l}(\mathsf{X}_0)$ associated with the unlabeled process X.

Let $\mathbf{X}^m=\{(X_t^1,\ldots,X_t^m)\}$ and $\mathsf{X}^{m*}=\{\sum_{m<j}\delta_{X_t^j}\}$. We then make the m-labeled process $(\mathbf{X}^m,\mathsf{X}^{m*})$ of the fully labeled process $\mathbf{X}$. It is known [30] that $(\mathbf{X}^m,\mathsf{X}^{m*})$ is a diffusion associated with a closable form $\mathcal{E}^{\mu^{[m]}}$ with domain $C_0^\infty(S^m)\otimes\mathcal{D}_\circ$ on $L^2(S^m\times\mathsf{S},\mu^{[1]})$ defined by

$$\mathcal{E}^{a,\mu^{[m]}}(f,g)=\int_{S^m\times\mathsf{S}}\left\{\frac{1}{2}\sum_{j=1}^m(a(x,\mathsf{s})\frac{\partial f}{\partial x_j},\frac{\partial g}{\partial x_j})_{\mathbb{R}^d}+\mathbb{D}^a[f,g]\right\}\mu^{[m]}(d\mathbf{x}_md\mathsf{s}). \tag{8.1}$$

Here $\frac{\partial}{\partial x_j}=(\frac{\partial}{\partial x_{j1}},\ldots,\frac{\partial}{\partial x_{jd}})$ is the nabla in $\mathbb{R}^d$ and $a(x,\mathsf{s})$ is given by (2.13). In this sense, it was proved that there exist the natural couplings among the Dirichlet spaces $(\mathcal{E}^{a,\mu^{[m]}},\mathcal{D}^{a,\mu^{[m]}})$ on $L^2(S^m\times\mathsf{S},\mu^{[m]})$ in [30].

In particular, each 1-labeled process $(X_t^i,\mathsf{X}_t^{i\diamondsuit})$ is associated with the one labeled Dirichlet space $(\mathcal{E}^{a,\mu^{[1]}},\mathcal{D}^{a,\mu^{[1]}})$ on $L^2(S\times\mathsf{S},\mu^{[1]})$. Let $\mathrm{Cap}^{a,\mu^{[1]}}$ denote its capacity.

The advantage of introducing the m-labeled processes is that one can regard the m particles $\mathbf{X}^m$ as a Dirichlet process of the diffusion associated with the Dirichlet form $(\mathcal{E}^{a,\mu^{[1]}},\mathcal{D}^{a,\mu^{[1]}})$. Namely, one can regard $A_t^{\mathbf{x_m}}:=\mathbf{X}_t^m-\mathbf{X}_0^m$ as an dm-dimensional additive functional generated by the composition of $(\mathbf{X}^m,\mathsf{X}^{m*})$ with the coordinate function $\mathbf{x}_m=(x_1,\ldots,x_m)\in(\mathbb{R}^d)^m$. Although $\mathbf{X}^m$ can be regarded as an additive functional of unlabeled process $\mathsf{X}=\sum_i\delta_{X^i}$, $\mathbf{X}^m$ is no longer a Dirichlet process in this case. Indeed, as a function of $\mathsf{X}=\{\mathsf{X}_s\}$, $\mathbf{X}_t^m$ can not be identified without tracing up all the trajectory of $\mathsf{X}_s=\sum_i\delta_{X_s^i}$ until $s\le t$.

Since $\mathbf{X}^m$ is a Dirichlet process, we can use Itô formula (Fukushima decomposition), and Lyons-Zheng decomposition to $\mathbf{X}^m$, which is a key to the proof of the results in this section.

8.1 Sufficient conditions of (A4) and (A5).

Let $\mathsf{S}_{\mathrm{s}} = \{\mathsf{s}\,;\, \mathsf{s}(\{x\}) \le 1 \text{ for all } x \in S\}$ be the set of all configurations with no multiple points. We quote a sufficient condition of

$$\mathrm{Cap}^{a,\mu}(\mathsf{S}_{\mathrm{s}}^c) = 0. \tag{8.2}$$

Here $\mathrm{Cap}^{a,\mu}$ is the capacity of the Dirichlet form $(\mathcal{E}^{a,\mu}, \mathcal{D}^{a,\mu})$ on $L^2(\mathsf{S},\mu)$ as before.

Lemma 8.1 ([29, Theorem 2.1, Proposition 7.1])**.** Assume **(A2)** and **(A3)**. Assume that Φ and Ψ are locally bounded from below. Then we obtain the following.
(1) Assume that $d \ge 2$ or that μ is a determinantal random point field with locally Lipschitz continuous kernel with respect to the Lebesgue measure. Then (8.2) holds.
(2) Assume (2.58) and (2.59). Then (8.2) holds.

Proof. (1) follows from [29, Theorem 2.1, Proposition 7.1] and (8.2). We can prove (2) in a similar fashion of the proof of [29, Theorem 2.1]. In fact, we easily deduce from the argument of [14] that [29, (4.5)] holds under the assumption (2.58) and (2.59), and the rest of the proof is exactly the same as [29, Theorem 2.1]. □

Proposition 8.2. Let the same assumptions of Lemma 8.1 hold. Assume **(A5')**. Then:
(1) Each tagged particle never explodes. Namely,

$$\mathsf{P}_{\mathsf{s}}(\sup_{t \le T} \{|X_t^i|\} < \infty \ \text{ for all } T, i \in \mathbb{N}\,) = 1 \quad \text{ for q.e. } \mathsf{s}. \tag{8.3}$$

(2) **(A4)** and **(A5)** hold.

Proof. Applying [9, Theorem 5.7.3] to $\mathbf{X}^m$ of this Dirichlet form, we see that the diffusion $(\mathbf{X}^m, \mathsf{X}^{m*})$ is conservative. Since this holds for all $m \in \mathbb{N}$, we obtain (8.3).

We next proceed with the proof of (2). We write $\mathsf{X}_t = \sum_i \delta_{X_t^i}$ such that $X^i \in C(I^i; S)$ as in (2.25). We will prove $I^i = [0,\infty)$ a.s. P_{s}. Recall that $I^i = [0,b)$ or $I^i = (a,b)$ for some $a, b \in (0,\infty]$. Suppose that $I^i = [0,b)$. Then from (8.3) we deduce that $b = \infty$. Next suppose that $I^i = (a,b)$. Then applying the strong Markov property of the diffusion $\{\mathsf{P}_{\mathsf{s}}\}$ at any $a' \in (a,b)$ and using the preceding argument, we deduce that $b = \infty$. As a result, we have $I^i = (a,\infty)$. Because of the reversibility, we see that $a = 0$, which yields contradiction. Hence we obtain $I^i = [0,\infty)$ for all i. This implies **(A5)**.

Recall that we have already obtained (8.2) from Lemma 8.1. Since $\mu(\{\mathsf{s}(S)=\infty\})=1$, **(A4)** follows from (8.2) and **(A5)**. Hence we see that **(A4)** is satisfied. We thus complete the proof of (2). □

8.2 Sufficient conditions of (A6).

Proposition 8.3. Assume that (8.4) below holds for all $r, T \in \mathbb{N}$. Then **(A6)** holds.

$$\sum_{i=1}^{\infty} \mathsf{P}_{\mu}(\sup_{t\in[0,T]} |M_t^i| \ge |s_i| - r) < \infty. \tag{8.4}$$

Here $\mathfrak{l}(\mathsf{s}) = (s_i)_{i=1}^{\infty} = \mathbf{s}$ in (2.7), and M^i is the continuous local martingale given by

$$M_t^i = \int_0^t \sigma(X_t^i, \mathsf{X}_t^{i\diamondsuit}) dB_t^i. \tag{8.5}$$

Proof. We use a Lyons-Zheng type decomposition [9]. Let $\mathbf{X} = (X^i)_{i\in\mathbb{N}}$ be a solution of (2.5) starting at $\mathbf{s} = (s_i)_{i\in\mathbb{N}}$. Namely, $X_0^i = s_i$ by definition. Note that M^i is the martingale part of the solution X^i in (2.5). Let $T \in \mathbb{N}$ be fixed and set for $0 \le t \le T$

$$\hat{M}_t^i = M_{T-t}^i(r_T) - M_T^i(r_T).$$

Here $r_T : C([0,T];\mathsf{S}) \to C([0,T];\mathsf{S})$ such that $r_T(w)(t) = w(T-t)$. Then by Lyons-Zheng type decomposition we have for each $0 \le t \le T$

$$X_t^i - X_0^i = \frac{1}{2} M_t^i + \frac{1}{2} \hat{M}_t^i \quad \text{for all } i \in \mathbb{N} \text{ under } \mathsf{P}_{\mu}. \tag{8.6}$$

Recall that $X_0^i = s_i$. Then we see that

$$\begin{aligned}
&\mathsf{P}_{\mu}(|X_t^i| \le r \text{ for some } t \in [0,T]) \\
\le &\mathsf{P}_{\mu}(|X_t^i - s_i| \ge |s_i| - r \text{ for some } t \in [0,T]) \\
\le &\mathsf{P}_{\mu}(\sup_{t\in[0,T]} |M_t^i| \ge |s_i| - r) + \mathsf{P}_{\mu}(\sup_{t\in[0,T]} |\hat{M}_t^i| \ge |s_i| - r) \\
= &2\mathsf{P}_{\mu}(\sup_{t\in[0,T]} |M_t^i| \ge |s_i| - r).
\end{aligned}$$

Hence from this and (8.4) we obtain that

$$\sum_{i=1}^{\infty} \mathsf{P}_{\mu}(|X_t^i| \le r \text{ for some } t \in [0,T]) \le 2\sum_{i=1}^{\infty} \mathsf{P}_{\mu}(\sup_{t\in[0,T]} |M_t^i| \ge |s_i| - r) < \infty. \tag{8.7}$$

Then we deduce **(A6)** from (8.7) and Borel-Cantelli's lemma. □

We can easily check (8.4) for all the examples in Section 2 because M^i are standard Brownian motions B^i of $\mathbb{R}^d$ in these examples. Indeed, we have the following.

Proposition 8.4. Suppose that $\sigma = \sqrt{c_{10}}E$. Let $\mathcal{N}$ be as in (2.26). Then **(A6)** follows from

$$\int_S \mathcal{N}(\frac{|x|-r}{\sqrt{c_{10}T}})\rho^1(x)dx < \infty. \tag{8.8}$$

Proof. Since $M_t^i = \sqrt{c_{10}}B_t^i$ and B_t^i is rotation invariant, we obtain for $|s_i| > r$

$$\mathsf{P}_\mu(\inf_{t\in[0,T]}(M_t^i, \frac{s_i}{|s_i|})_{\mathbb{R}^d} \le r - |s_i|) = \mathsf{P}_\mu(\sup_{t\in[0,T]}(B_t^i, \frac{s_i}{|s_i|})_{\mathbb{R}^d} \ge |s_i| - r)2\int_{\mathsf{S}} \mathcal{N}(|s_i|-r)\mathsf{P}_\mu(d\mathsf{s}). \tag{8.9}$$

Here we used the fact that $(B_t^i, \frac{s_i}{|s_i|})_{\mathbb{R}^d}$ is the standard 1D Brownian motion and its reflection principle. From the standard calculation of correlation functions and (8.8) we deduce that

$$\sum_{i=1}^\infty \int_{\mathsf{S}} \mathcal{N}(\frac{|s_i|-r}{\sqrt{c_{10}T}})\mathsf{P}_\mu(d\mathsf{s}) = \int_{\mathsf{S}} \sum_{i=1}^\infty \mathcal{N}(\frac{|s_i|-r}{\sqrt{c_{10}T}})\mathsf{P}_\mu(d\mathsf{s}) = \int_S \mathcal{N}(\frac{|x|-r}{\sqrt{c_{10}T}})\rho^1(x)dx < \infty. \tag{8.10}$$

Hence we deduce **(A6)** from (8.9) and (8.10) immediately. □

8.3 Sufficient conditions of (A7) and (A8).

Let $\mathbf{a} = \{a_\mathsf{k}\}_{\mathsf{k}\in\mathbb{N}}$ be a sequence of increasing sequences $a_\mathsf{k} = \{a_\mathsf{k}(r)\}_{r\in\mathbb{N}}$ of natural numbers such that $a_\mathsf{k}(r) < a_{\mathsf{k}+1}(r)$ for all $r, \mathsf{k} \in \mathbb{N}$. We set for $\mathbf{a} = \{a_\mathsf{k}\}_{\mathsf{k}\in\mathbb{N}}$ that

$$\mathsf{K}[\mathbf{a}] = \bigcup_{\mathsf{k}=1}^\infty \mathsf{K}[a_\mathsf{k}], \quad \mathsf{K}[a_\mathsf{k}] = \{\mathsf{s}\,;\, \mathsf{s}(S_r) \le a_\mathsf{k}(r) \text{ for all } r \in \mathbb{N}\}. \tag{8.11}$$

By construction $\mathsf{K}[a_\mathsf{k}] \subset \mathsf{K}[a_{\mathsf{k}+1}]$ for all $\mathsf{k} \in \mathbb{N}$. It is well known that $\mathsf{K}[a_\mathsf{k}]$ is a compact set in S for each $\mathsf{k} \in \mathbb{N}$. We assume:

(U1) The sequence $\mathbf{a}$ satisfies

$$\mu(\mathsf{K}[\mathbf{a}]) = 1. \tag{8.12}$$

We remark that, when μ is a translation invariant random point field on $\mathbb{R}^d$, we can take $a_\mathsf{k}(r) = \mathsf{k}r^d$. We also remark that a sequence $\mathbf{a}$ satisfying (8.12) always exists for any random point filed μ (see [28, Lemma 2.6]).

Let $\mathsf{S}_{\text{s.i.}}^{[1]} = \{(x,\mathsf{s}) \in S \times \mathsf{S}\,;\, \delta_x + \mathsf{s} \in \mathsf{S}_{\text{s.i.}}\}$. Let $a_\mathsf{k}^+(r) = a_\mathsf{k}(r+1)$ and $\mathbf{a}^+ = \{a_\mathsf{k}^+\}_{\mathsf{k}\in\mathbb{N}}$. Let

$$\mathsf{H}[\mathbf{a}]_{\mathsf{p},\mathsf{q},\mathsf{k}} = \{(x,\mathsf{s}) \in \mathsf{S}_{\text{s.i.}}^{[1]}\,;\, \inf_i |x - s_i| \ge 2^{-\mathsf{p}},\ x \in S_\mathsf{q},\ \mathsf{s} \in \mathsf{K}[a_\mathsf{k}^+]\}, \quad (\mathsf{s} = \sum_i \delta_{s_i}), \tag{8.13}$$

$$\mathsf{H}[\mathbf{a}] = \bigcup_{\mathsf{p},\mathsf{q},\mathsf{k}=1}^\infty \mathsf{H}[\mathbf{a}]_{\mathsf{p},\mathsf{q},\mathsf{k}}. \tag{8.14}$$

Lemma 8.5. Assume **(A3)**, **(A4')** and **(U1)**. Then the following holds.

$$\mathrm{Cap}^{\mu^{[1]}}(\mathsf{H}[\mathbf{a}]^c) = 0. \tag{8.15}$$

Proof. The claim (8.15) follows from **(A4')** and

$$\mathrm{Cap}^{\mu}(\mathsf{K}[\mathbf{a}^{+}]^{c}) = 0. \tag{8.16}$$

If $\sigma_r^k \in L^{\infty}(S_r^k, d\mathbf{x}^k)$, then from this, **(U1)**, and **(A2)**, we deduce that the assumptions (A1) and (A2) of [28] are fulfilled. Hence (8.16) follows from [28, Lemma 2.5 (4)] easily. In the present paper, we assume that $\sigma_r^k \in L^{p}(S_r^k, d\mathbf{x}^k)$ $(1 < p \le \infty)$ in **(A3)** instead. The assumption $\sigma_r^k \in L^{\infty}(S_r^k, d\mathbf{x}^k)$ was used in [28] only in the proof of [28, Lemma 2.4]. By replacing $C_r^k = \|\sigma_r^k\|_{L^{\infty}(Q_r^k, dx)}$ by $C_r^k = \|\sigma_r^k\|_{L^{p}(Q_r^k, dx)}$ in the last line in [28, 125 p.], and applying the Hölder inequality to the inequality in the second line in [28, 126 p.], we obtain the same conclusion as [28, Lemma 2.4]. As a result, we obtain that of [28, Lemma 2.5 (4)], which yields (8.16). ☐

The next lemma is clear by construction.

Lemma 8.6. Let $\mathsf{H}[\mathbf{a}]_{\mathsf{p},\mathsf{q},\mathsf{k}}$ be as in (8.13). Let $(\mathsf{p}(\mathsf{n}), \mathsf{q}(\mathsf{n}), \mathsf{k}(\mathsf{n}))$ be an increasing sub sequence of $(\mathsf{p}, \mathsf{q}, k)$ such that $\lim \mathsf{p}(\mathsf{n}) = \lim \mathsf{q}(\mathsf{n}) = \lim \mathsf{k}(\mathsf{n}) = \infty$ as $\mathsf{n} \to \infty$. Set

$$\mathsf{H}[\mathbf{a}]_{\mathsf{n}} = \mathsf{H}[\mathbf{a}]_{\mathsf{p}(\mathsf{n}),\mathsf{q}(\mathsf{n}),\mathsf{k}(\mathsf{n})}.$$

Then $\mathsf{H}[\mathbf{a}]_{\mathsf{n}} \subset \mathsf{H}[\mathbf{a}]_{\mathsf{n}+1}$ and $\bigcup_{n=1}^{\infty} \mathsf{H}[\mathbf{a}]_{\mathsf{n}} = \mathsf{H}[\mathbf{a}]$.

Let (σ, b) be the coefficients of the ISDE (2.5) and set $\mathbb{F}^{[0]} = \{\hat{\sigma}, \hat{b}\}$. Here $(\hat{\sigma}, \hat{b})$ is a version of (σ, b). Then we assume:

(U2) There exist a subset $\mathsf{I}[\mathbf{a}]$ and a version $(\hat{\sigma}, \hat{b})$ of (σ, b) such that

$$c_{11} = \sup\{\frac{|f(x,\mathsf{s}) - f(x',\mathsf{s})|}{|x - x'|};\, x \neq x',\, (x,\mathsf{s}), (x',\mathsf{s}) \in \mathsf{H}[\mathbf{a}]_{\mathsf{n}} \cap \mathsf{I}[\mathbf{a}],\ f \in \mathbb{F}^{[0]}\} < \infty, \tag{8.17}$$

$$c_{12} = \sup\{|f(x,\mathsf{s})|;\, (x,\mathsf{s}) \in \mathsf{H}[\mathbf{a}]_{\mathsf{n}} \cap \mathsf{I}[\mathbf{a}],\ f \in \mathbb{F}^{[0]}\} < \infty, \tag{8.18}$$

$$\mathrm{Cap}^{\mu^{[1]}}(\mathsf{I}[\mathbf{a}]^{c}) = 0. \tag{8.19}$$

Here $c_{11} = c_{11}(\mathsf{n})$ and $c_{12} = c_{12}(\mathsf{n})$ are constants depending on $\mathsf{n} \in \mathbb{N}$.

The assumption **(U2)** implies that each $f \in \mathbb{F}^{[0]}$ is bounded and Lipschitz continuous in x on $\mathsf{H}[\mathbf{a}]_{\mathsf{n}} \cap \mathsf{I}[\mathbf{a}]$ uniformly in s for all $\mathsf{n} \in \mathbb{N}$. Furthermore, we deduce from Lemma 8.5 and (8.19) that

$$\mathrm{Cap}^{\mu^{[1]}}(\{\mathsf{H}[\mathbf{a}] \cap \mathsf{I}[\mathbf{a}]\}^{c}) = 0. \tag{8.20}$$

Proposition 8.7. Assume **(A3)**, **(A4')**, **(U1)**, and **(U2)**. Then **(A7)** and **(A8)** hold with $\mathsf{H}[\mathbf{a}] \cap \mathsf{I}[\mathbf{a}]$.

Proof. Let $(\mathbf{Z}, \mathsf{Z}) \in C([0,\infty); S^m \times \mathsf{S})$, and we write $\mathbf{Z} = (Z^1, \ldots, Z^m)$ and $\mathsf{Z} = \sum_{j>m}^{\infty} \delta_{Z^j}$. Let $\zeta_{\mathsf{n}} = \zeta_{\mathsf{n}}(\mathbf{Z}, \mathsf{Z})$ be functions on the m-labeled path space $C([0,\infty); S^m \times \mathsf{S})$ defined by

$$\zeta_{\mathsf{n}}(\mathbf{Z}, \mathsf{Z}) = \inf\{t > 0\,;\, (Z_t^i, \sum_{j\neq i}^{m} \delta_{Z_t^j} + \mathsf{Z}_t) \notin \mathsf{H}[\mathbf{a}]_{\mathsf{n}} \cap \mathsf{I}[\mathbf{a}] \text{ for some } i = 1, \ldots, m\,\}. \tag{8.21}$$

Note that $\zeta_{\mathsf{n}-1}(\mathbf{Z},\mathsf{Z}) \le \zeta_{\mathsf{n}}(\mathbf{Z},\mathsf{Z})$ by Lemma 8.6.

Let m be fixed and set $\mathbf{s}_m = (s_1,\ldots,s_m)$. Let $\mathbf{Y} = (Y^{m,1},\ldots,Y^{m,m})$ be a solution of (2.31) with (2.32). Let $\mathsf{q} \ge 1+\max\{|s_i|;\, i=1,\ldots,m\}$ and $\mathsf{X}^{m*} = \sum_{j=m+1}\delta_{X_t^j}$. By **(U2)** the SDE (2.31) has a strong solution and the pathwise uniqueness holds until some of the (S,S)-valued processes

$$(Y_t^{m,i}, \sum_{j\neq i}^{m} \delta_{Y_t^{m,j}} + \mathsf{X}_t^{m*}) \quad (i=1,\ldots,m) \tag{8.22}$$

exit from $\mathsf{H}[\mathbf{a}]_{\mathsf{n}}$. Hence, because of the pathwise uniqueness, the solution $(\mathbf{Y}_t^m,\mathbf{B}_t^m)$ coincides with $(\mathbf{X}_t^m,\mathbf{B}_t^m)$ until $t<\zeta_{\mathsf{n}}(\mathbf{Y}^m.\mathsf{X}^{m*})$. This coincidence also implies that $\zeta_{\mathsf{n}}(\mathbf{Y}^m.\mathsf{X}^{m*}) = \zeta_{\mathsf{n}}(\mathbf{X}^m.\mathsf{X}^{m*})$. In particular, $\{\mathbf{Y}_t^m\}$ and $\{\mathbf{X}_t^m\}$ are functions of the Brownian motion $\mathbf{B}^m$ and X^{m*} until $0 \le t < \zeta_{\mathsf{n}}(\mathbf{X}^m.\mathsf{X}^{m*})$. Hence we write $\mathbf{Y}_t^m(\mathbf{B}^m,\mathsf{X}^{m*})$ and $\mathbf{X}_t^m(\mathbf{B}^m,\mathsf{X}^{m*})$. Then above mentioned coincidence implies, for all $\mathsf{n}\in\mathbb{N}$,

$$\mathbf{Y}_t^m(\mathbf{B}^m,\mathsf{X}^{m*}) = \mathbf{X}_t^m(\mathbf{B}^m,\mathsf{X}^{m*}) \quad \text{ for each } t<\zeta_{\mathsf{n}}. \tag{8.23}$$

We deduce from (8.20) that $\lim_{\mathsf{n}\to\infty}\zeta_{\mathsf{n}}(\mathbf{X}^m.\mathsf{X}^{m*}) = \infty$. Hence the equality in (8.23) holds for all $t<\lim_{\mathsf{n}\to\infty}\zeta_{\mathsf{n}}=\infty$. This completes the proof. □

8.4 A sufficient condition of (U2).

In this subsection we give a sufficient condition of **(U2)**. For this we begin by introducing the cut off functions on $S\times\mathsf{S}$. Let $h:\mathbb{R}\to\mathbb{R}$ such that

$$h_{\mathsf{p}}(t) = \begin{cases} 1 & t\le 0, \\ 1-2^{\mathsf{p}+1}t & (0\le t\le 2^{-\mathsf{p}-1}), \\ 0 & (2^{-\mathsf{p}-1}\le t). \end{cases} \tag{8.24}$$

Let $d_{\mathsf{p}}:S\times\mathsf{S}\to\mathbb{R}$ such that

$$d_{\mathsf{p}}(x,\mathsf{s}) = \{\sum_{|x-s_i|\le 2^{-\mathsf{p}}} (2^{-\mathsf{p}} - |x-s_i|)^2\}^{1/2}, \quad \text{ where } \mathsf{s} = \sum_i \delta_{s_i}. \tag{8.25}$$

Let $\varphi_{\mathsf{q}}\in C_0^\infty(S)$ such that $\varphi_{\mathsf{q}}(x) = \varphi_{\mathsf{q}}(|x|)$, $0\le\varphi_{\mathsf{q}}(x)\le 1$ for all $x\in S$, $\varphi_{\mathsf{q}}(x)=1$ for $x\in S_{\mathsf{q}}$, $\varphi_{\mathsf{q}}(x)=0$ for $x\in S_{\mathsf{q}+1}^c$, and $|\nabla\varphi_{\mathsf{q}}(x)|\le 2$ for all $x\in S$. Let

$$\chi_{a_{\mathsf{k}}}(\mathsf{s}) = h_0\circ d_{a_{\mathsf{k}}}(\mathsf{s}), \quad d_{a_{\mathsf{k}}}(\mathsf{s}) = \{\sum_{r=1}^{\infty}\sum_{j\in J(\mathsf{k},r,\mathsf{s})} (r-|s_j|)^2\}^{1/2}. \tag{8.26}$$

Here we label $\mathsf{s}=\sum_i\delta_{s_i}$ in such a way that $|s_i|\le|s_{i+1}|$ for all i, and we set

$$J(\mathsf{k},r,\mathsf{s}) = \{j\,;\, j>a_{\mathsf{k}}(r),\ s_j\in S_r\}. \tag{8.27}$$

Then we introduce the cut-off functions $\chi_{\mathsf{p},\mathsf{q},\mathsf{k}}$ and $\chi_{\mathsf{p},\infty}$ defined by

$$\chi_{\mathsf{p},\mathsf{q},\mathsf{k}}(x,\mathsf{s}) = h_{\mathsf{p}}(d_{\mathsf{p}}(x,\mathsf{s}))\varphi_{\mathsf{q}}(x)\chi_{a_{\mathsf{k}}}(\mathsf{s}), \quad \chi_{\mathsf{p},\infty}(x,\mathsf{s}) = h_{\mathsf{p}}(d_{\mathsf{p}}(x,\mathsf{s}))\chi_{a_{\mathsf{k}}}(\mathsf{s}). \tag{8.28}$$

Let $(\mathsf{p}(\mathsf{n}),\mathsf{q}(\mathsf{n}),\mathsf{k}(\mathsf{n}))$ be the subsequence of $(\mathsf{p},\mathsf{q},\mathsf{k})$ in Lemma 8.6. We set $\chi_{\mathsf{n}} = \chi_{\mathsf{p}(\mathsf{n}),\mathsf{q}(\mathsf{n}),\mathsf{k}(\mathsf{n})}$. Then $\{\chi_{\mathsf{n}}\}$ are consistent in the sense that $\chi_{\mathsf{n}}(x,\mathsf{s}) = \chi_{\mathsf{n}+1}(x,\mathsf{s})$ for $(x,\mathsf{s}) \in \mathsf{H}[\mathbf{a}]_{\mathsf{n}}$. Hence we see that

$$\lim_{\mathsf{n}\to\infty} \chi_{\mathsf{n}}(x,\mathsf{s}) = 1 \quad \text{for all } (x,\mathsf{s}) \in \mathsf{H}[\mathbf{a}]. \tag{8.29}$$

We will localize functions on $S \times \mathsf{S}$ supported on $\mathsf{H}[\mathbf{a}]$ by using χ_{n} $(\mathsf{n} \in \mathbb{N})$.

[][] 計算 By a direct calculation similar to [28, Lemma 2.5], we obtain the following.

Lemma 8.8. The functions χ_{n} $(\mathsf{n} \in \mathbb{N})$ satisfy the following:

$$\chi_{\mathsf{n}}(x,\mathsf{s}) = 0 \quad \text{for } (x,\mathsf{s}) \notin \mathsf{H}[\mathbf{a}]_{\mathsf{n}+1}, \tag{8.30}$$

$$\chi_{\mathsf{n}} \in \mathcal{D}^{\mu^{[1]}}, \tag{8.31}$$

$$|\chi_{\mathsf{n}}(x,\mathsf{s})| \le 1, \quad |\nabla\chi_{\mathsf{n}}(x,\mathsf{s})|^2 \le c_{13}, \quad \mathbb{D}[\chi_{\mathsf{n}},\chi_{\mathsf{n}}](x,\mathsf{s}) \le c_{14}. \tag{8.32}$$

Here $c_{13} = c_{13}(\mathsf{n})$ and $c_{14} = c_{14}(\mathsf{n})$ are positive constants independent of (x,s).

Let $\mathcal{D}^{\mu^{[1]}}$ be the domain of the Dirichlet form of the one labeled process $(X^i, \mathsf{X}^{i\diamondsuit})$. Let

$$\mathbf{J}^{[l]} = \{(j_1,\dots,j_d);\ 0 \le j_i \le l,\ \sum_{i=1}^d j_i = l\}.$$

We assume that there exists an $\ell \in \{0\} \cup \mathbb{N}$ satisfying the following:

(Z1) For each $\mathbf{j} \in \cup_{l=1}^{\ell-1}\mathbf{J}^{[l]}$, it holds that $\chi_{\mathsf{n}}\partial_{\mathbf{j}}\sigma, \chi_{\mathsf{n}}\partial_{\mathbf{j}}b \in \mathcal{D}^{\mu^{[1]}}$ for all $\mathsf{n} \in \mathbb{N}$.

(Z2) For each $\mathbf{j} \in \mathbf{J}^{[\ell]}$, there exist $\mathsf{g}_{\mathbf{j}}$ and $\mathsf{h}_{\mathbf{j}}$ satisfying

$$\sum_i |\mathsf{g}_{\mathbf{j}}(x,s_i)|,\ \sum_i |\mathsf{h}_{\mathbf{j}}(x,s_i)| \in L^\infty(\mathsf{H}[\mathbf{a}]_{\mathsf{n}}, \mu^{[1]}) \quad \text{for all } \mathsf{n} \in \mathbb{N}, \tag{8.33}$$

$$\partial_{\mathbf{j}}\sigma(x,\mathsf{s}) = \sum_i \mathsf{g}_{\mathbf{j}}(x-s_i), \quad \partial_{\mathbf{j}}b(x,\mathsf{s}) = \sum_i \mathsf{h}_{\mathbf{j}}(x-s_i) \quad \text{for } (x,\mathsf{s}) \in \mathsf{H}[\mathbf{a}]. \tag{8.34}$$

Here $\mathsf{s} = \sum_i \delta_{s_i}$ as before, and $\mathsf{H}[\mathbf{a}]_{\mathsf{n}}$ and $\mathsf{H}[\mathbf{a}]$ are as in Lemma 8.6.

Lemma 8.9. Assume **(Z1)** and **(Z2)**. Then **(U2)** holds.

Proof. From **(Z1)** and **(Z2)** combined with the Lebesgue's convergence theorem we see that for any $\mathbf{j}' \in \mathbf{J}^{[\ell-1]}$, the functions $\partial_{\mathbf{j}'}\sigma(x,\mathsf{s})$ and $\partial_{\mathbf{j}'}b(x,\mathsf{s})$ are bounded, Lipschitz continuous on $\mathsf{H}[\mathbf{a}]_{\mathsf{n}}$, and (8.33) and (8.34) with replacement of $\mathbf{j}$ by $\mathbf{j}'$ also hold. Repeating this in l inductively from $l = \ell - 1$ to $l = 0$ yields the claim. ∎

The following sufficient condition of (8.33) is useful for Examples in Section 2.

Lemma 8.10. We assume that $\mathbf{a} = \{a_{\mathsf{k}}(r)\}$, $\mathsf{g}_{\mathbf{j}}$, and $\mathsf{h}_{\mathbf{j}}$ satisfy for each $\mathsf{n} \in \mathbb{N}$

$$\sum_{r=1}^{\infty} \frac{a_{\mathsf{k}(\mathsf{n})}(r)}{r^{c_{16}}} < \infty, \tag{8.35}$$

$$|\mathsf{g}_{\mathbf{j}}(x,s)|,\ |\mathsf{h}_{\mathbf{j}}(x,s)| \le c_{15}(1+|s|)^{-c_{16}} \quad \text{for all } (x,s) \in \mathsf{H}[\mathbf{a}]_{\mathsf{n}},\ \mathbf{j} \in \mathbf{J}^{[\ell]}. \tag{8.36}$$

Here $c_{15} = c_{15}(\mathsf{n})$ and $c_{16} = c_{16}(\mathsf{n})$ are positive constants. Then (8.33) holds.

Proof. From (8.35) and (8.36) we deduce that

$$\sum_{\mathsf{K}[a_{\mathsf{k}(\mathsf{n})}]} |\mathsf{g}_{\mathbf{j}}(x,s_i)| \le \sum_{r=1}^{\infty} \sum_{\substack{\mathsf{K}[a_{\mathsf{k}(\mathsf{n})}] \\ s_i \in S_r \setminus S_{r-1}}} \frac{c_{15}}{(1+|s_i|)^{c_{16}}} \le \sum_{r=1}^{\infty} \frac{c_{15} a_{\mathsf{k}(\mathsf{n})}(r)}{r^{c_{16}}}. \tag{8.37}$$

This yields (8.33) for $\mathsf{g}_{\mathbf{j}}$, and the proof for $\mathsf{h}_{\mathbf{j}}$ is the same. □

9 Proof of Theorem 2.3.

The purpose of this section is to prove Theorem 2.3. Let $\mu_{\mathrm{sin},\beta}$ be the Sine$_\beta$ random point fields ($\beta = 1, 2, 4$). We will check that $\mu_{\mathrm{sin},\beta}$ satisfy the assumptions **(A1)**–**(A8)**.

We begin by checking **(A1)**. The following is essentially obtained in [31, Theorem 82].

Lemma 9.1. (1) The logarithmic derivative $\mathsf{d}^{\mu_{\mathrm{sin},\beta}}$ of $\mu_{\mathrm{sin},\beta}$ exists in $L^2_{\mathrm{loc}}(S \times \mathsf{S}, \mu^{[1]}_{\mathrm{sin},\beta})$.
(2) The logarithmic derivative $\mathsf{d}^{\mu_{\mathrm{sin},\beta}}$ has expressions:

$$\mathsf{d}^{\mu_{\mathrm{sin},\beta}}(x,\mathsf{s}) = \beta \lim_{r\to\infty} \sum_{|x-s_i|<r} \frac{1}{x-s_i}, \tag{9.1}$$

$$\mathsf{d}^{\mu_{\mathrm{sin},\beta}}(x,\mathsf{s}) = \beta \lim_{r\to\infty} \sum_{|s_i|<r} \frac{1}{x-s_i}. \tag{9.2}$$

Proof. The existence of the logarithmic derivative $\mathsf{d}^{\mu_{\mathrm{sin},\beta}}$ of the form (9.1) in $L^p_{\mathrm{loc}}(S \times \mathsf{S}, \mu^{[1]}_{\mathrm{sin},\beta})$ for any $1 \le p < 2$ is obtained in [31, Theorem 82]. Since the labeled density of $\mu^{[1]}_{\mathrm{sin},\beta}$ around $\{(x,\mathsf{s}); |x-s_i| = 0\}$ is of order $|x-s_i|^2$, we can refine this as in (9.1) and (9.2).

Let $\Delta_r = \Delta_r^1 \cup \Delta_r^2$, where $\Delta_r^1 = \{|s_i| < r\} \cap \{|x-s_i| < r\}^c$ and $\Delta_r^2 = \{|s_i| < r\}^c \cap \{|x-s_i| < r\}$. Since $d = 1$, we easily see from the estimates of one and two points correlation functions that

$$\lim_{r\to\infty} \sum_{\Delta_r} \frac{1}{x-s_i} = 0 \quad \text{in } L^2_{\mathrm{loc}}(S \times \mathsf{S}, \mu^{[1]}_{\mathrm{sin},\beta}). \tag{9.3}$$

Hence (9.2) follows from (9.1). □

Lemma 9.2. **(A2)**–**(A6)** hold.

Proof. In [32, Theorem 2.2], it was proved that $\mu_{\sin,\beta}$ is a $(0,-\beta\log|x-y|)$-quasi-Gibbs measure for $\beta=1,2,4$. This yields **(A2)**. **(A3)** is immediate from (2.44) for $\beta=2$, and a similar determinantal expression of correlation functions in [24] for $\beta=1,4$. **(A4)** and **(A5)** follow from Proposition 8.2. **(A6)** follows from Proposition 8.4. □

We next proceed with the proof of **(A7)** and **(A8)**. For this we use Proposition 8.7 and Lemma 8.9. Let $\mathbf{a}=\{a_{\mathsf{k}}\}$ be as in (8.11). Since $\mu_{\sin,\beta}$ are translation invariant, we can take $a_{\mathsf{k}}(r)=\mathsf{k}r$. This implies **(U1)**. To check **(U2)**, it suffices to prove **(Z1)** and **(Z2)**. We take $\ell=0$.

Lemma 9.3. For any $\mathsf{n}\in\mathbb{N}$

$$\sum_i \frac{1}{|x-s_i|^2} \in L^{\infty}(\mathsf{H}[\mathbf{a}]_{\mathsf{n}}, \mu_{\mathrm{Sin},\beta}^{[1]}), \tag{9.4}$$

and $\nabla \mathsf{d}^{\mu_{\sin,\beta}}$ is given by

$$\nabla \mathsf{d}^{\mu_{\sin,\beta}} = -\beta \lim_{r\to\infty} \sum_{|x-s_i|<r} \frac{1}{|x-s_i|^2}. \tag{9.5}$$

Proof. The assumptions of Lemma 8.10 is satisfied with $c_{16}=3/2$, which yields (9.4). We can check (9.5) from (9.4) and Lemma 9.1 through the Lebesgue convergence theorem. □

Lemma 9.4. Let $\mathcal{D}_{\mathrm{Sin},\beta}^{[1]}$ be the domain of the one labeled process of Sine_β IBMs ($\beta=1,2,4$). Then

$$\chi_{\mathsf{n}}\mathsf{d}^{\mu_{\sin,\beta}} \in \mathcal{D}_{\mathrm{Sin},\beta}^{[1]} \quad \text{for all } \mathsf{n}\in\mathbb{N}. \tag{9.6}$$

Proof. By definition

$$\mathcal{E}^{\mu_{\mathrm{Sin},\beta}^{[1]}}(\chi_{\mathsf{n}}\mathsf{d}^{\mu_{\sin,\beta}}, \chi_{\mathsf{n}}\mathsf{d}^{\mu_{\sin,\beta}}) = \int_{S\times\mathsf{S}} \frac{1}{2}|\nabla\chi_{\mathsf{n}}\mathsf{d}^{\mu_{\sin,\beta}}|^2 + \mathbb{D}[\chi_{\mathsf{n}}\mathsf{d}^{\mu_{\sin,\beta}}, \chi_{\mathsf{n}}\mathsf{d}^{\mu_{\sin,\beta}}] d\mu_{\mathrm{Sin},\beta}^{[1]}. \tag{9.7}$$

From Lemma 8.8, Lemma 9.1 (1), and Lemma 9.3, we deduce that

$$\int_{S\times\mathsf{S}} |\nabla\chi_{\mathsf{n}}\mathsf{d}^{\mu_{\sin,\beta}}|^2 d\mu_{\mathrm{Sin},\beta}^{[1]} \le 2\int_{\mathsf{H}[\mathbf{a}]_{\mathsf{n}+1}} \{\chi_{\mathsf{n}}^2|\nabla\mathsf{d}^{\mu_{\sin,\beta}}|^2 + |\nabla\chi_{\mathsf{n}}|^2|\mathsf{d}^{\mu_{\sin,\beta}}|^2\} d\mu_{\mathrm{Sin},\beta}^{[1]} < \infty. \tag{9.8}$$

From the Schwarz inequality and Lemma 8.8, we deduce that

$$\begin{aligned}
&\int_{S\times\mathsf{S}} \mathbb{D}[\chi_{\mathsf{n}}\mathsf{d}^{\mu_{\sin,\beta}}, \chi_{\mathsf{n}}\mathsf{d}^{\mu_{\sin,\beta}}] d\mu_{\mathrm{Sin},\beta}^{[1]} \\
&\le 2\int_{S\times\mathsf{S}} \chi_{\mathsf{n}}^2\mathbb{D}[\mathsf{d}^{\mu_{\sin,\beta}}, \mathsf{d}^{\mu_{\sin,\beta}}] + \mathbb{D}[\chi_{\mathsf{n}},\chi_{\mathsf{n}}]\,|\mathsf{d}^{\mu_{\sin,\beta}}|^2\, d\mu_{\mathrm{Sin},\beta}^{[1]} \\
&\le 2\int_{\mathsf{H}[\mathbf{a}]_{\mathsf{n}+1}} \mathbb{D}[\mathsf{d}^{\mu_{\sin,\beta}}, \mathsf{d}^{\mu_{\sin,\beta}}] + c_{14}|\mathsf{d}^{\mu_{\sin,\beta}}|^2\, d\mu_{\mathrm{Sin},\beta}^{[1]} \\
&\le 2\int_{\mathsf{H}[\mathbf{a}]_{\mathsf{n}+1}} \frac{\beta^2}{2}\Big(\sum_{\frac{1}{\mathsf{p}(\mathsf{n}+1)}\le|x-s_i|<\infty} \frac{1}{|x-s_i|^4}\Big) + c_{14}|\mathsf{d}^{\mu_{\sin,\beta}}|^2\, d\mu_{\mathrm{Sin},\beta}^{[1]} \\
&< \infty.
\end{aligned} \tag{9.9}$$

Here the last line follows from a direct calculation and Lemma 9.1.

We note that $\chi_{\mathsf{n}}\mathsf{d}^{\mu_{\sin,\beta}} \in L^2(S \times \mathsf{S}, \mu_{\mathrm{Sin},\beta}^{[1]})$ is obvious from Lemma 9.1. Hence combining (9.7), (9.8), and (9.9), we obtain that $\chi_{\mathsf{n}}\mathsf{d}^{\mu_{\sin,\beta}} \in \mathcal{D}_{\mathrm{Sin},\beta}^{[1]}$. □

Lemma 9.5. The coefficients of ISDEs (2.43) satisfy **(A7)** and **(A8)**.

Proof. Note that $b(x,\mathsf{y}) = \frac{\beta}{2}\mathsf{d}^{\mu_{\sin,\beta}}$. Then from Lemma 9.4 we deduce **(U2)**. Hence we obtain **(A7)** and **(A8)** from Proposition 8.7. □

Proof of Theorem 2.3. **(A1)**–**(A8)** follow from Lemma 9.1, Lemma 9.2, and Lemma 9.5. Hence we obtain Theorem 2.3 from Theorem 2.2. □

10 Proof of Theorem 2.5

In this section we prove Theorem 2.5.

Let μ_{Gin} be the Ginibre random point field. The following is essentially obtained in [31].

Lemma 10.1. (1) The logarithmic derivative $\mathsf{d}^{\mu_{\mathrm{Gin}}}$ of μ_{Gin} exists in $L^2_{\mathrm{loc}}(S \times \mathsf{S}, \mu_{\mathrm{Gin}}^{[1]})$.
(2) The logarithmic derivative $\mathsf{d}^{\mu_{\mathrm{Gin}}}$ has plural expressions:

$$\mathsf{d}^{\mu_{\mathrm{Gin}}}(x,\mathsf{s}) = 2 \lim_{r\to\infty} \sum_{|x-s_i|<r} \frac{x-s_i}{|x-s_i|^2}, \tag{10.1}$$

$$\mathsf{d}^{\mu_{\mathrm{Gin}}}(x,\mathsf{s}) = -2x + 2 \lim_{r\to\infty} \sum_{|s_i|<r} \frac{x-s_i}{|x-s_i|^2}. \tag{10.2}$$

Here the convergence takes place in $L^2_{\mathrm{loc}}(S \times \mathsf{S}, \mu_{\mathrm{Gin}}^{[1]})$.

Proof. The existence of the logarithmic derivative $\mathsf{d}^{\mu_{\mathrm{Gin}}}$ of the form (10.1) and (10.2) for any $1 \le p < 2$ is obtained in [31, Theorem 61, Lemma 72]. We refine this as follows: In [31, Lemma 72], the convergence in $L^2_{\mathrm{loc}}(S \times \mathsf{S}, \mu_{\mathrm{Gin}}^{[1]})$ of the series

$$\mathsf{d}_1^{\mu_{\mathrm{Gin}}}(x,\mathsf{s}) := 2 \lim_{r\to\infty} \sum_{1\le|x-s_i|<r} \frac{x-s_i}{|x-s_i|^2}, \tag{10.3}$$

$$\mathsf{d}_2^{\mu_{\mathrm{Gin}}}(x,\mathsf{s}) := -2x + 2 \lim_{r\to\infty} \sum_{1\le|s_i|<r} \frac{x-s_i}{|x-s_i|^2} \tag{10.4}$$

is proved. Since the labeled density of $\mu_{\mathrm{Gin}}^{[1]}$ around $\{(x,\mathsf{s}); |x-s_i| = 0\}$ is of order $|x-s_i|^2$, we deduce that

$$\mathsf{d}_3^{\mu_{\mathrm{Gin}}}(x,\mathsf{s}) := \lim_{r\to\infty} 2 \sum_{|x-s_i|<1} \frac{x-s_i}{|x-s_i|^2} \in L^2_{\mathrm{loc}}(S \times \mathsf{S}, \mu_{\mathrm{Gin}}^{[1]}), \tag{10.5}$$

$$\mathsf{d}_4^{\mu_{\mathrm{Gin}}}(x,\mathsf{s}) := -2x + 2 \sum_{|s_i|<1} \frac{x-s_i}{|x-s_i|^2} \in L^2_{\mathrm{loc}}(S \times \mathsf{S}, \mu_{\mathrm{Gin}}^{[1]}). \tag{10.6}$$

Since $\mathsf{d}^{\mu_{\mathrm{Gin}}} = \mathsf{d}_1^{\mu_{\mathrm{Gin}}} + \mathsf{d}_2^{\mu_{\mathrm{Gin}}} = \mathsf{d}_3^{\mu_{\mathrm{Gin}}} + \mathsf{d}_4^{\mu_{\mathrm{Gin}}}$, we conclude Theorem 10.1. □

Lemma 10.2. **(A2)**–**(A6)** hold.

Proof. In [32, Theorem 2.3], it was proved that μ_{Gin} is a $(|x|^2, -2\log|x-y|)$-quasi-Gibbs measure. This yields **(A2)**. **(A3)** is immediate from (2.53). **(A4)** and **(A5)** follow from Proposition 8.2. **(A6)** follows from Proposition 8.4. □

We next proceed with the proof of **(A7)** and **(A8)**. For this we use Proposition 8.7 and Lemma 8.9. Let $\mathbf{a} = \{a_{\mathsf{k}}\}$ be as in (8.11). Since μ_{Gin} is translation invariant, we can take $a_{\mathsf{k}}(r) = \mathsf{k}r^2$. This implies **(U1)**. To check **(U2)**, it suffices to prove **(Z1)** and **(Z2)**. We take $\ell = 2$. We prepare the following.

Lemma 10.3. Let $A:\mathbb{R}^2\to\mathbb{R}^4$ be the 2×2 matrix valued function defined by

$$A(x) = \begin{pmatrix} (y-z)^2 & -2yz \\ -2yz & (y-z)^2 \end{pmatrix} \quad \text{for } x=(y,z)\in\mathbb{R}^2. \tag{10.7}$$

Let $\mathsf{H}[\mathbf{a}]_{\mathsf{n}}$ and φ_r be as in Subsections 8.3 and 8.4. Then for any $\mathsf{n}\in\mathbb{N}$

$$\lim_{r\to\infty} 2\sum_i \varphi_r(|x-s_i|)\frac{A(x-s_i)}{|x-s_i|^4} \quad \text{in } L^2(\mathsf{H}[\mathbf{a}]_{\mathsf{n}}, \mu_{\mathrm{Gin}}^{[1]}). \tag{10.8}$$

Furthermore, $\nabla\mathsf{d}^{\mu_{\mathrm{Gin}}} \in L^2(\mathsf{H}[\mathbf{a}]_{\mathsf{n}}, \mu_{\mathrm{Gin}}^{[1]})$ for all $\mathsf{n}\in\mathbb{N}$, and $\nabla\mathsf{d}^{\mu_{\mathrm{Gin}}}$ is given by

$$\nabla\mathsf{d}^{\mu_{\mathrm{Gin}}} = 2\lim_{r\to\infty}\sum_i \varphi_r(|x-s_i|)\frac{A(x-s_i)}{|x-s_i|^4}. \tag{10.9}$$

Proof. Let

$$\mathsf{d}_r^{\mu_{\mathrm{Gin}}}(x,\mathsf{s}) = 2\sum_i \varphi_r(|x-s_i|)\frac{x-s_i}{|x-s_i|^2}.$$

Then from (10.1) we can easily deduce that, for each $\mathsf{n}\in\mathbb{N}$,

$$\mathsf{d}^{\mu_{\mathrm{Gin}}}(x,\mathsf{s}) = \lim_{r\to\infty}\mathsf{d}_r^{\mu_{\mathrm{Gin}}}(x,\mathsf{s}) \quad \text{in } L^2(\mathsf{H}[\mathbf{a}]_{\mathsf{n}}, \mu_{\mathrm{Gin}}^{[1]}). \tag{10.10}$$

By a direct calculation we see that

$$\nabla\mathsf{d}_r^{\mu_{\mathrm{Gin}}}(x,\mathsf{s}) = 2\sum_i \nabla\varphi_r(|x-s_i|)\otimes\frac{x-s_i}{|x-s_i|^2} + \varphi_r(|x-s_i|)\frac{A(x-s_i)}{|x-s_i|^4}. \tag{10.11}$$

It is not difficult to see that

$$\lim_{r\to\infty} 2\sum_i \nabla\varphi_r(|x-s_i|)\otimes\frac{x-s_i}{|x-s_i|^2} = 0. \tag{10.12}$$

Let $\varphi_{r,s} = \varphi_s - \varphi_r$ for $r<s$ and set

$$G_{r,s} = 2\chi_{\mathsf{n}}\sum_i \varphi_{r,s}(|x-s_i|)\frac{A(x-s_i)}{|x-s_i|^4} \tag{10.13}$$

$$H_{r,s} = 2\chi_{\mathsf{n}}\sum_i \varphi_{r,s}(|x-s_i|)\frac{A(x-s_i)}{|x-s_i|^4}\lceil|x-s_i|\rceil. \tag{10.14}$$

Here $\lceil a \rceil$ is the minimal integer greater than or equal to a. Then we see that

$$G_{r,s} = \frac{H_{r,s}}{s} + \sum_{t=r+1}^{s-1} \frac{H_{r,t}}{t(t+1)}. \tag{10.15}$$

Note that $E^{\mu_{\mathrm{Gin}}^{[1]}}[H_{r,t}] = 0$. Applying [36, Theorem 1.3], we see that $\mathrm{Var}^{\mu_{\mathrm{Gin}}^{[1]}}[H_{r,t}] = O(t)$. Hence the series in (10.15) converge in $L^2(\mathsf{H}[\mathbf{a}]_{\mathsf{n}}, \mu_{\mathrm{Gin}}^{[1]})$ for each $r \in \mathbb{N}$, and $\lim_{s\to\infty} H_{r,s}/s = 0$. We thus have

$$\lim_{s\to\infty} G_{r,s} = \sum_{t=r+1}^{\infty} \frac{H_{r,t}}{t(t+1)} \quad \text{in } L^2(\mathsf{H}[\mathbf{a}]_{\mathsf{n}}, \mu_{\mathrm{Gin}}^{[1]}). \tag{10.16}$$

From (10.11)–(10.13), and (10.16), we complete the proof of Lemma 10.3. □

Lemma 10.4. Let $\mathcal{D}_{\mathrm{Gin}}^{[1]}$ be the domain of the one labeled process of Ginibre IBMs. Then

$$\chi_{\mathsf{n}} \mathsf{d}^{\mu_{\mathrm{Gin}}},\ \chi_{\mathsf{n}} \nabla \mathsf{d}^{\mu_{\mathrm{Gin}}} \in \mathcal{D}_{\mathrm{Gin}}^{[1]} \quad \text{for all } \mathsf{n} \in \mathbb{N}. \tag{10.17}$$

Proof. We only prove $\chi_{\mathsf{n}} \mathsf{d}^{\mu_{\mathrm{Gin}}} \in \mathcal{D}_{\mathrm{Gin}}^{[1]}$ because the proof of $\chi_{\mathsf{n}} \nabla \mathsf{d}^{\mu_{\mathrm{Gin}}} \in \mathcal{D}_{\mathrm{Gin}}^{[1]}$ is easier than the previous one. By definition

$$\mathcal{E}^{\mu_{\mathrm{Gin}}^{[1]}}(\chi_{\mathsf{n}} \mathsf{d}^{\mu_{\mathrm{Gin}}}, \chi_{\mathsf{n}} \mathsf{d}^{\mu_{\mathrm{Gin}}}) = \int_{S\times\mathsf{S}} \frac{1}{2} |\nabla \chi_{\mathsf{n}} \mathsf{d}^{\mu_{\mathrm{Gin}}}|^2 + \mathbb{D}[\chi_{\mathsf{n}} \mathsf{d}^{\mu_{\mathrm{Gin}}}, \chi_{\mathsf{n}} \mathsf{d}^{\mu_{\mathrm{Gin}}}] d\mu_{\mathrm{Gin}}^{[1]}. \tag{10.18}$$

From $\chi_{\mathsf{n}}(x) = 0$ for $x \in \mathsf{H}[\mathbf{a}]_{\mathsf{n}+1}^c$, Lemma 8.8, Lemma 10.1 (1), and Lemma 10.3, we deduce that

$$\int_{S\times\mathsf{S}} |\nabla \chi_{\mathsf{n}} \mathsf{d}^{\mu_{\mathrm{Gin}}}|^2 d\mu_{\mathrm{Gin}}^{[1]} \le 2 \int_{\mathsf{H}[\mathbf{a}]_{\mathsf{n}+1}} \{\chi_{\mathsf{n}}^2 |\nabla \mathsf{d}^{\mu_{\mathrm{Gin}}}|^2 + |\nabla \chi_{\mathsf{n}}|^2 |\mathsf{d}^{\mu_{\mathrm{Gin}}}|^2\} d\mu_{\mathrm{Gin}}^{[1]} < \infty. \tag{10.19}$$

From Schwarz inequality and Lemma 8.8, we deduce that

$$\begin{aligned} &\int_{S\times\mathsf{S}} \mathbb{D}[\chi_{\mathsf{n}} \mathsf{d}^{\mu_{\mathrm{Gin}}}, \chi_{\mathsf{n}} \mathsf{d}^{\mu_{\mathrm{Gin}}}] d\mu_{\mathrm{Gin}}^{[1]} \\ \le\, & 2 \int_{S\times\mathsf{S}} \left\{ \chi_{\mathsf{n}}^2 \mathbb{D}[\mathsf{d}^{\mu_{\mathrm{Gin}}}, \mathsf{d}^{\mu_{\mathrm{Gin}}}] + \mathbb{D}[\chi_{\mathsf{n}}, \chi_{\mathsf{n}}]\, |\mathsf{d}^{\mu_{\mathrm{Gin}}}|^2 \right\} d\mu_{\mathrm{Gin}}^{[1]} \\ \le\, & 2 \int_{\mathsf{H}[\mathbf{a}]_{\mathsf{n}+1}} \left\{ \mathbb{D}[\mathsf{d}^{\mu_{\mathrm{Gin}}}, \mathsf{d}^{\mu_{\mathrm{Gin}}}] + c_{14}^2 |\mathsf{d}^{\mu_{\mathrm{Gin}}}|^2 \right\} d\mu_{\mathrm{Gin}}^{[1]} \\ \le\, & 2 \int_{\mathsf{H}[\mathbf{a}]_{\mathsf{n}+1}} \left\{ c_{17} \sum_{\frac{1}{\mathsf{p}(\mathsf{n}+1)} \le |x-s_i| < \infty} \frac{1}{|x-s_i|^4} + c_{14}^2 |\mathsf{d}^{\mu_{\mathrm{Gin}}}|^2 \right\} d\mu_{\mathrm{Gin}}^{[1]} \\ <\, & \infty. \end{aligned} \tag{10.20}$$

Here c_{17} is a positive constant such that

$$\mathbb{D}[\mathsf{d}^{\mu_{\mathrm{Gin}}}, \mathsf{d}^{\mu_{\mathrm{Gin}}}](x, \mathsf{s}) \le c_{17} \sum_i \frac{1}{|x-s_i|^4} \quad \text{for all } (x, \mathsf{s}).$$

We easily see that $c_{17} < \infty$, and that the last line in (10.20) follows from this, a direct calculation and Lemma 10.1.

We note that $\chi_{\mathsf{n}} \mathsf{d}^{\mu_{\mathrm{Gin}}} \in L^2(S \times \mathsf{S}, \mu_{\mathrm{Gin}}^{[1]})$ is obvious from Lemma 10.1. Hence combining (10.18), (10.19), and (10.20), we obtain that $\chi_{\mathsf{n}} \mathsf{d}^{\mu_{\mathrm{Gin}}} \in \mathcal{D}_{\mathrm{Gin}}^{[1]}$. □

Lemma 10.5. The coefficients of ISDEs (2.51) and (2.52) satisfy **(A7)** and **(A8)**.

Proof. We only prove the case of (2.51) because that of (2.52) is similar. To prove (2.51) we will use Proposition 8.7. For this our task is to check **(U1)** and **(U2)**.

We have already taken $a_{\mathsf{k}}(r) = \mathsf{k}r^2$ and $\ell = 2$. Then **(U1)** is clear from $a_{\mathsf{k}}(r) = \mathsf{k}r^2$ because of the translation invariance of μ_{Gin}.

We next check **(U2)** by using Lemma 8.9. The assumption **(Z1)** follows from Lemma 10.4. By definition σ is the 2×2 unit matrix. Hence $\nabla\sigma = 0$, which combined with (2.12) implies $b(x, \mathsf{y}) = \frac{1}{2}\mathsf{d}^{\mu_{\mathrm{Gin}}}$. We take $\mathsf{g}_{\mathbf{j}}(x, s) = 0$ and $\mathsf{h}_{\mathbf{j}}(x, s) = \partial_{\mathbf{j}} \frac{x-s}{|x-s|^2}$ for $\mathbf{j} \in \mathbf{J}^{[2]}$. We can easily check that the assumptions (8.35) and (8.36) of Lemma 8.10 are fulfilled. Hence we deduce (8.33) from Lemma 8.10. Furthermore, (8.34) clearly holds. Hence we obtain **(U2)** from Lemma 8.9.

We therefore conclude **(A7)** and **(A8)** from Proposition 8.7. □

Proof of Theorem 2.5. **(A1)**–**(A8)** follow from Lemma 10.1, Lemma 10.2, and Lemma 10.5. Hence we obtain (1) and (2) from Theorem 2.2. (3) follows from Lemma 10.1. □

11 Proof of Theorem 2.6.

The purpose of this subsection is to prove Theorem 2.6.

Let μ_{Ψ_0} be as in Theorem 2.6. We will check that μ_{Ψ_0} satisfy all the assumptions **(A1)**–**(A8)**.

We begin by the calculation of the logarithmic derivative.

Lemma 11.1. **(A1)** holds. The logarithmic derivative $\mathsf{d}^{\mu_{\Psi_0}}$ is given by

$$\mathsf{d}^{\mu_{\Psi_0}}(x, \mathsf{y}) = -\sum_i \nabla\Psi_0(x - y_i). \tag{11.1}$$

Proof. This lemma is clear from DLR equation. Indeed, we can see this as follows.

Let $\mathsf{S}_r^{[1],m} = S_r \times \mathsf{S}_{r,m}$ for $m = 0, 1, \ldots$. Below we consider the case only for $m \neq 0$.

Let $\mu_{r,\eta}^{[1]}$ be the regular conditional probability of the 1-Campbell measure $\mu_{\Psi_0}^{[1]}$ conditioned at $\pi_{S_r^c}(\mathsf{y}) = \pi_{S_r^c}(\eta)$ for $(x, \mathsf{y}) \in S \times \mathsf{S}$. Let $\sigma_{r,\eta}^{[1],m}$ be the density function of $\mu_{r,\eta}^{[1]}$ on $\mathsf{S}_r^{[1],m}$ with respect to the Lebesgue measure. Then by DLR equation and the definitions of Palm and Campbell measures we can see by a direct calculation that

$$\sigma_{r,\eta}^{[1],m}(x, \mathbf{y}_m) = \frac{1}{\mathcal{Z}_{r,\eta}^m} e^{-\sum_{i=1}^m \Psi_0(x-y_i) - \sum_{i<j}^m \Psi_0(y_i - y_j) - \sum_{\eta_k \in S_r^c} \{\Psi_0(x-\eta_k) + \sum_{i=1}^m \Psi_0(y_i - \eta_k)\}}. \tag{11.2}$$

Hence we see that

$$\nabla_x \log \sigma_{r,\eta}^{[1],m}(x, \mathbf{y}_m) = -\sum_{i=1}^m \nabla_x \Psi_0(x - y_i) - \sum_{\eta_k \in S_r^c} \nabla_x \Psi_0(x - \eta_i). \tag{11.3}$$

Since, for $\varphi \in C_0(S) \times \mathcal{D}_\circ$ such that $\varphi = 0$ on S_r^c, we have that

$$
\begin{aligned}
(11.4)\qquad & - \int_{S\times\mathsf{S}} \nabla_x \varphi(x,\mathbf{y}) \mu_{\Psi_0}^{[1]} = - \int_{S_r\times\mathsf{S}} \nabla_x \varphi \, \mu_{r,\eta}^{[1]}(dxd\mathsf{y}) \, \mu_{\Psi_0}^{[1]} \circ \pi_{S_r^c}^{-1}(d\eta) \\
&= - \int_{S_r\times\mathsf{S}} \nabla_x \varphi \, \sigma_{r,\eta}^{[1],m}(x,\mathbf{y}) \, dxd\mathbf{y} \, \mu_{\Psi_0}^{[1]} \circ \pi_{S_r^c}^{-1}(d\eta) \\
&= \int_{S_r\times\mathsf{S}} \varphi \, \nabla_x \log \sigma_{r,\eta}^{[1],m}(x,\mathbf{y}) \, dxd\mathbf{y} \, \mu_{\Psi_0}^{[1]} \circ \pi_{S_r^c}^{-1}(d\eta)
\end{aligned}
$$

From (11.3) and (11.4) we obtain (11.1). □

Lemma 11.2. **(A2)**–**(A6)** hold.

Proof. **(A2)** is obvious from the DLR equation and the assumption that Ψ_0 is smooth outside the origin. Since Ψ_0 is super stable, Ψ_0 is bounded from below. From this **(A3)** is clear because the local densities given by the DLR equation become bounded from above. **(A4)** and **(A5)** follow from Proposition 8.2. **(A6)** follows from Proposition 8.4. □

Lemma 11.3. **(A7)** and **(A8)** hold.

Proof. To prove Theorem 11.2 it suffices to check **(U1)** and **(U2)**. **(U1)** is obvious by assumption. To prove **(U2)** we check **(Z1)** and **(Z2)**. We take $\ell = 0$. Then **(Z1)** and (8.33) are clear by assumption. (8.34) follows from Lemma 8.10. □

Proof of Theorem 2.6. We have thus checked from Lemma 11.1, Lemma 11.2, and Lemma 11.3 that μ_{Ψ_0} satisfies all the assumptions **(A1)**–**(A8)**. Hence Theorem 2.6 follows from Theorem 2.2. □

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3

[3] The proof of uniqueness follows from Theorem 3.4 (2) and (2.35).

In fact, we take $\mathcal{G}$ in Proposition 5.7 [1:56] as

[:R1k]
$$\mathcal{G} = \sigma[H]. \tag{11.5}$$

Then $\mathcal{G}$ is countably determined for any families of probability measures on $\mathcal{G}$ because $\mathcal{L}$ is a Polish space and H is a measurable function.

Let $\mathbf{s} \in \mathbf{S}_0$ be fixed. Let $\mathbf{Y_s} = \mathbf{Y_s}(\mathbf{B})$ and $\mathbf{Y}'_\mathbf{s} = \mathbf{Y}'_\mathbf{s}(\mathbf{B})$ be two strong solutions defined on the same Brownian motion $\mathbf{B}$. Let $\bar{P}_\mathbf{s}$ and $\bar{P}'_\mathbf{s}$ be the distributions of $(\mathbf{X_s}, \mathbf{B})$ and $(\mathbf{X}'_\mathbf{s}, \mathbf{B})$.

Then by construction $\bar{P}_{\mathbf{s},\mathbf{B}} = \delta_{\mathbf{Y_s}(\mathbf{B})}$ and $\bar{P}'_{\mathbf{s},\mathbf{B}} = \delta_{\mathbf{Y}'_\mathbf{s}(\mathbf{B})}$.

多分不要