Gauss 膜モデルの漸近挙動について

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内容

- モデルの定義と基本的性質
- ピンニングポテンシャルによる局在化
- エントロピー的反発

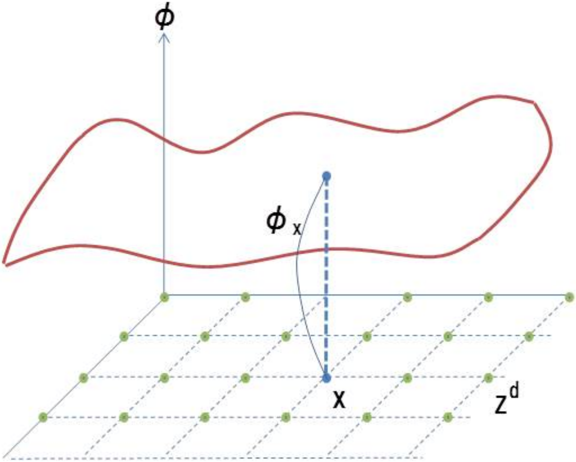
Gaussian membrane model ($\Delta\phi$ model)

- $\phi = \{\phi_x\}_{x \in \mathbb{Z}^d} \in \mathbb{R}^{\mathbb{Z}^d}$

  : $d + 1$ - dimensional random membrane/interface

- $x \in \mathbb{Z}^d \mapsto \phi_x \in \mathbb{R}$: height at the position x

Gaussian membrane model ($\Delta\phi$ model)



Gaussian membrane model ($\Delta\phi$ model)

- $\phi = \{\phi_x\}_{x \in \mathbb{Z}^d} \in \mathbb{R}^{\mathbb{Z}^d}$

  : $d+1$ - dimensional random membrane/interface

- $x \in \mathbb{Z}^d \mapsto \phi_x \in \mathbb{R}$: height at the position x

- $H(\phi) := \frac{1}{2} \sum_{x \in \mathbb{Z}^d} (\Delta\phi_x)^2$: energy of ϕ

  where

  $\Delta f(x) = \sum_{y \in \mathbb{Z}^d} \Delta(x, y) f(y)$ for $f : \mathbb{Z}^d \to \mathbb{R}$

$$\Delta(x,y) = \begin{cases} \frac{1}{2d} & \text{if } |x-y| = 1 \\ -1 & \text{if } x = y \\ 0 & \text{otherwise} \end{cases}$$

Gaussian membrane model ($\Delta\phi$ model)

Notation

For $\Lambda \subset \mathbb{Z}^d$ and $k \in \mathbb{N}$,

$$\partial_k^+ \Lambda := \{x \notin \Lambda; \|y - x\|_1 \leq k \text{ for some } y \in \Lambda\}$$
$$\partial_k^- \Lambda := \{x \in \Lambda; \|y - x\|_1 \leq k \text{ for some } y \notin \Lambda\}$$
$$\partial^+ \Lambda := \partial_1^+ \Lambda, \ \partial^- \Lambda := \partial_1^- \Lambda, \ \overline{\Lambda} := \Lambda \cup \partial^+ \Lambda$$

For $\Lambda \Subset \mathbb{Z}^d$, define

$$P_\Lambda(d\phi) = \frac{1}{Z_\Lambda} \exp\{-H_\Lambda^\Delta(\phi)\} \prod_{x \in \Lambda} d\phi_x$$

where

$$H_\Lambda^\Delta(\phi) = \frac{1}{2} \sum_{x \in \mathbb{Z}^d} (\Delta \phi_x)^2 \Big|_{\phi \equiv 0 \text{ on } \Lambda^c} = \frac{1}{2} \sum_{x \in \overline{\Lambda}} (\Delta \phi_x)^2 \Big|_{\phi \equiv 0 \text{ on } \partial_2^+ \Lambda}$$

$$Z_\Lambda = \int_{\mathbb{R}^\Lambda} \exp\{-H_\Lambda^\Delta(\phi)\} \prod_{x \in \Lambda} d\phi_x$$

Gaussian membrane model ($\Delta\phi$ model)

Remark 1

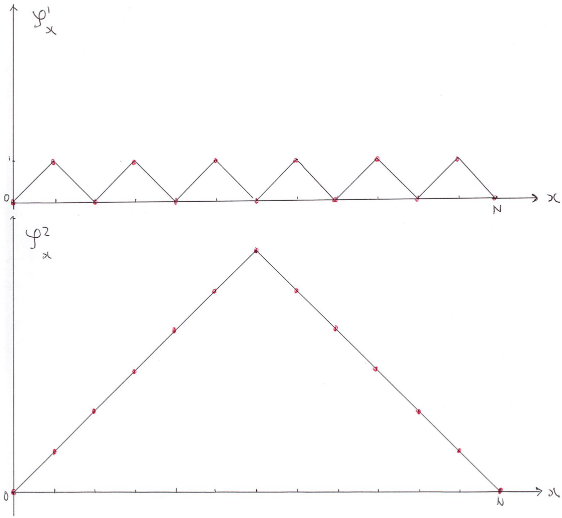
Model for semi-flexible membrane :

$$H(\phi) = \sum_{x \in \mathbb{Z}^d} \left(\kappa_1 (\nabla \phi_x)^2 + \kappa_2 (\Delta \phi_x)^2 \right)$$

where κ_1 : lateral tension κ_2 : bending rigidity

- $P_\Lambda \leftrightarrow$ (static) model of tensionless membrane
- $\kappa_1 > 0,\ \kappa_2 = 0 \quad \leftrightarrow \quad \nabla\phi$ interface model

Gaussian membrane model ($\Delta\phi$ model)



$$H^{\nabla}(\phi^1) = H^{\nabla}(\phi^2) \quad H^{\Delta}(\phi^1) \gg H^{\Delta}(\phi^2)$$

Basic properties

First of all, we have

$$
\begin{aligned}
H_\Lambda^\Delta(\phi) &= \frac{1}{2}\sum_{x\in\overline{\Lambda}} (\Delta\phi_x)^2\Big|_{\phi\equiv 0 \text{ on } \partial_2^+\Lambda} \\
&= \frac{1}{2}\sum_{x\in\overline{\Lambda}}\Big(\sum_{y\in\Lambda}\Delta(x,y)\phi_y\Big)\Big(\sum_{z\in\Lambda}\Delta(x,z)\phi_z\Big) \\
&= \frac{1}{2}\sum_{y\in\Lambda}\sum_{z\in\Lambda}\Delta^2(y,z)\phi_y\phi_z \\
&= \frac{1}{2}\langle\phi,\, \Delta_\Lambda^2\phi\rangle_\Lambda
\end{aligned}
$$

where

$$
\Delta_\Lambda^2(x,y) = \begin{cases} \Delta^2(x,y) & \text{if } x, y \in \Lambda \\ 0 & \text{otherwise} \end{cases}
$$

Basic properties

$$P_\Lambda(d\phi) = \frac{1}{Z_\Lambda} \exp\{-\frac{1}{2}\langle \phi, \Delta_\Lambda^2 \phi \rangle_\Lambda\} \prod_{x\in\Lambda} d\phi_x$$

$\Rightarrow P_\Lambda \sim \mathcal{N}(0, (\Delta_\Lambda^2)^{-1})$: Gaussian membrane model (GMM)

Remark 2 ($\nabla\phi$ model)

$$H_\Lambda^\nabla(\phi) = \frac{1}{8d} \sum_{\substack{\{x,y\}\cap\Lambda\neq\phi \\ |x-y|=1}} (\phi_x - \phi_y)^2 \Big|_{\phi\equiv 0 \text{ on } \partial^+\Lambda}$$

$$P_\Lambda^\nabla(d\phi) = \frac{1}{Z_\Lambda^\nabla} \exp\Big\{-H_\Lambda^\nabla(\phi)\Big\} \prod_{x\in\Lambda} d\phi_x$$

$$= \frac{1}{Z_\Lambda^\nabla} \exp\{-\frac{1}{2}\langle \phi, (-\Delta_\Lambda)\phi \rangle_\Lambda\} \prod_{x\in\Lambda} d\phi_x$$

$\Rightarrow P_\Lambda^\nabla \sim \mathcal{N}(0, (-\Delta_\Lambda)^{-1})$: DGFF on Λ

Basic properties

Remark 3

$$\Delta_\Lambda^2 \neq (\Delta_\Lambda)^2 \qquad ((\Delta_\Lambda)^2)^{-1} = ((-\Delta_\Lambda)^{-1})^2$$

Remark 4

$$\Delta^2(x,y) = \begin{cases} 1 + \frac{1}{2d} & \text{if } x = y \\ -\frac{1}{d} & \text{if } |x-y| = 1 \\ \frac{1}{4d^2} & \text{if } |x-y| = 2 \\ \frac{1}{2d^2} & \text{if } |x-y| = \sqrt{2} \\ 0 & \text{otherwise} \end{cases}$$

Basic properties

GMM has $\mathbf{2}$-step Markov property, i.e.

$\boldsymbol{A, B \subset \Lambda, \mathrm{dist}(A, B) > 2}$

$\Rightarrow \{\phi_x\}_{x \in A}, \{\phi_x\}_{x \in B}$ are independent

under $P_\Lambda(\,\cdot\,|\sigma(\{\phi_x; x \notin A \cup B\}))$

GMM is not a ferromagnetic spin system

$\Rightarrow$ $\times$ Random walk representation of the covariance

$\times$ Correlation inequalities (e.g. FKG), monotonicity

GMM is much less tractable compared to DGFF from the mathematical point of view !

Basic properties

Random walk representation for Gaussian fields

Consider a Gaussian field on $\Lambda (\Subset \mathbb{Z}^d)$

$$P_\Lambda(d\phi) = \frac{1}{Z_\Lambda} \exp\Big\{-\frac{1}{2}\langle \phi, J_\Lambda \phi\rangle_\Lambda\Big\} \prod_{x\in\Lambda} d\phi_x$$

For $J : \mathbb{Z}^d \times \mathbb{Z}^d \to \mathbb{R}$, assume the following :

- $J(x,y) = J(0, y-x)$ for every $x, y \in \mathbb{Z}^d$
- $J(x,y) \leq 0$ for every $x \neq y$
- $J(0,0) \geq -\sum_{y\neq 0} J(0,y) =: \gamma \ (\in (0,\infty))$

Set

$$Q(x,y) := \begin{cases} 0 & \text{if } x = y \\ -\frac{1}{\gamma} J(x,y) & \text{if } x \neq y \end{cases}$$

Basic properties

Then, we have

$$\langle \phi, J_\Lambda \phi \rangle_\Lambda = \sum_{x,y\in\Lambda} J(x,y)\phi_x\phi_y$$
$$= \sum_{x,y\in\Lambda} \big(J(0,0)\delta(x,y) - \gamma Q(x,y)\big)\phi_x\phi_y$$
$$= J(0,0)\Big\langle \phi, (I_\Lambda - \frac{\gamma}{J(0,0)} Q_\Lambda)\phi \Big\rangle_\Lambda$$

$$\big(I_\Lambda - \frac{\gamma}{J(0,0)} Q_\Lambda\big)^{-1} = \sum_{n\geq 0} \big(\frac{\gamma}{J(0,0)}\big)^n Q_\Lambda^n$$

$\leftrightarrow$ Green function of RW on $\mathbb{Z}^d$ with transition matrix Q, killing rate $1 - \frac{\gamma}{J(0,0)}$, Dirichlet boundary condition outside Λ

Basic properties

Example 1 (DGFF)

$$J = -\Delta \;\Rightarrow J(0,0) = \gamma := -\sum_{y\neq 0} J(0,y)$$

DGFF has a random walk representation of the covariance :

$$\begin{aligned} \mathrm{Cov}_{P^{\nabla}_{\Lambda}}(\phi_x\,,\phi_y) &= (-\Delta_\Lambda)^{-1}(x,y) \\ &= \mathbb{E}_x\Big[\sum_{n=0}^{\infty} I(S_n = y, n < \tau_\Lambda)\Big] \end{aligned}$$

where $\{S_n\}_{n\geq 0}$: simple random walk on $\mathbb{Z}^d$

$\tau_\Lambda = \inf\{n \geq 0; S_n \notin \Lambda\}$, $\Lambda \subset \mathbb{Z}^d$

Example 2 (massive GFF)

$$J = m^2 I - \Delta \;\left(H_\Lambda(\phi) = \tfrac{1}{2}\langle \phi, (-\Delta_\Lambda)\phi\rangle_\Lambda + \tfrac{1}{2}m^2\langle\phi,\phi\rangle_\Lambda\right)$$

$\Rightarrow$ SRW on $\mathbb{Z}^d$ with killing rate $\dfrac{m^2}{m^2+1}$

FKG inequality for random fields

Consider a probability measure on $\mathbb{R}^{\Lambda}$ ($\Lambda \Subset \mathbb{Z}^d$)

$$\mu_\Lambda(d\phi) = \frac{1}{Z_\Lambda} \exp\Big\{ -H_\Lambda(\phi) \Big\} \prod_{x \in \Lambda} d\phi_x$$

Proposition 1 (FKG inequality, Holley (1974) etc.)

If $H_\Lambda(\phi \vee \widetilde{\phi}) + H_\Lambda(\phi \wedge \widetilde{\phi}) \leq H_\Lambda(\phi) + H_\Lambda(\widetilde{\phi})$ for every $\phi, \widetilde{\phi} \in \mathbb{R}^{\Lambda}$ then μ_Λ satisfies FKG inequality i.e.,

$$\int_{\mathbb{R}^\Lambda} F(\phi) G(\phi) \mu_\Lambda(d\phi) \geq \int_{\mathbb{R}^\Lambda} F(\phi) \mu_\Lambda(d\phi) \int_{\mathbb{R}^\Lambda} G(\phi) \mu_\Lambda(d\phi)$$

for every non-decreasing functions $F, G \in L^2(\mathbb{R}^{\Lambda}, \mu_\Lambda)$

Basic properties

Example 1 ($\nabla\phi$ model)

For $H_\Lambda^{\nabla}(\phi) = \displaystyle\sum_{\substack{\{x,y\}\cap\Lambda\neq\phi \\ |x-y|=1}} V(\phi_x - \phi_y)\Big|_{\phi\equiv 0 \text{ on } \partial^+\Lambda}$

$$V(a-d) + V(b-c) \leq V(a-c) + V(b-d)$$

for every $a < b$, $c > d$ i.e., V : convex

$\Rightarrow$ the condition of Proposition 1 holds

Example 2 ($\Delta\phi$ model)

For $H_\Lambda^{\Delta}(\phi) = \displaystyle\sum_{x\in\overline{\Lambda}} V(\Delta\phi_x)\Big|_{\phi\equiv 0 \text{ on } \partial_2^+\Lambda}$

$\Rightarrow$ the condition of Proposition 1 does not hold for arbitrary V

Basic properties

Remark 5 (FKG inequality for P_{Λ}^{Δ})

- By Pitt (1982),
  FKG inequality holds for a Gaussian measure $\mathcal{N}(m, \Sigma)$.
  $\Leftrightarrow$ All elements of Σ are non-negative
- Inverse positivity of Δ_{Λ}^{2} is non-trivial and it depends on the underlying set Λ
- By RW representation, All elements of
  $((\Delta_{\Lambda})^{2})^{-1} = (-\Delta_{\Lambda})^{-1}(-\Delta_{\Lambda})^{-1}$ are non-negative
  $\Rightarrow$ FKG inequality holds for $\mathcal{N}(0, (\Delta_{\Lambda})^{-2})$

Variance of GMM

In the case of $\Lambda = \Lambda_N := [-N, N]^d \cap \mathbb{Z}^d$,
we denote $\Delta_{\Lambda_N}, P_{\Lambda_N}$ as Δ_N, P_N etc..

$$(-\Delta_N)^{-1}(0,x) = \mathbb{E}_0\Big[\sum_{n=0}^{\infty} I(S_n = x, n < \tau_\Lambda)\Big]$$

$$= \begin{cases} N + 1 - |x| & \text{if } d = 1 \\ C_2(\log N - \log(|x|+1)) + O(|x|^{-1}) & \text{if } d = 2 \\ C_d(|x|^{2-d} - N^{2-d}) + O(|x|^{1-d}) & \text{if } d \geq 3 \end{cases}$$

When $d \geq 3$,

$$(-\Delta_N)^{-2}(0,0) = \sum_{x \in \Lambda_N} ((-\Delta_N)^{-1}(0,x))^2$$

$$\asymp C \sum_{r=1}^{N} r^{d-1}\Big(\frac{1}{r^{d-2}}\Big)^2 = C \sum_{r=1}^{N} r^{-d+3}$$

Basic properties

Lemma 1 (Random walk representation for $(-\Delta_N)^{-2}$)

$$(-\Delta_N)^{-2}(x,y) = \sum_{n=0}^{\infty} (n+1)\mathbb{P}_x\big(S_n = y, n < \tau_{\Lambda_N}\big)$$

Proof.

$$\begin{aligned}
&(-\Delta_N)^{-2}(x,y)\\
&= \sum_{z\in\Lambda_N} \Big\{\sum_{n=0}^{\infty} \mathbb{P}_x\big(S_n = z, n < \tau_\Lambda\big)\Big\}\Big\{\sum_{k=0}^{\infty} \mathbb{P}_z\big(S_k = y, k < \tau_\Lambda\big)\Big\}\\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \mathbb{P}_x\big(S_{n+k} = y, n+k < \tau_\Lambda\big)\\
&= \sum_{n=0}^{\infty} (n+1)\mathbb{P}_x\big(S_n = y, n < \tau_\Lambda\big)
\end{aligned}$$

□

Basic properties

Proposition 2 (Comparsion between $(-\Delta_N)^{-2}$ and $(\Delta_N^2)^{-1}$)

Let $0 < \delta < 1$. There exists a constant $C = C(\delta) > 0$ such that

$$\max_{y \in \Lambda_{\delta N}} \left| (-\Delta_N)^{-2}(x, y) - (\Delta_N^2)^{-1}(x, y) \right| \leq C N^{4-d}$$

for every $x \in \Lambda_{\delta N}$

By comparing $(\Delta_N^2)^{-1}$ with $(-\Delta_N)^{-2}$, the following holds.

Proposition 3 (Variance of GMM)

There exist constants $\gamma_d > 0$ such that

$$\begin{aligned} \mathrm{Var}_{P_N^{\Delta}}(\phi_0) &= (\Delta_N^2)^{-1}(0, 0) \\ &= \begin{cases} \gamma_d + O(N^{4-d}) & \text{if } d \geq 5 \\ \gamma_d \log N + O(1) & \text{if } d = 4 \end{cases} \end{aligned}$$

$$\frac{1}{\gamma_d} N^{4-d} \leq \mathrm{Var}_{P_N^{\Delta}}(\phi_0) \leq \gamma_d N^{4-d} \quad \text{if } d \leq 3$$

Basic properties

Remark 6 (Variance of DGFF)

There exist constants $\gamma_d' > 0$ such that

$$\begin{aligned} \mathrm{Var}_{P_N^\nabla}(\phi_0) &= (-\Delta_N)^{-1}(0,0) \\ &= \begin{cases} \gamma_d' + O(N^{2-d}) & \text{if } d \geq 3 \\ \gamma_d' \log N + O(1) & \text{if } d = 2 \\ \gamma_d' N(1 + o(1)) & \text{if } d = 1 \end{cases} \end{aligned}$$

When $d \geq 3$, asymptotics of the variance of GMM corresponds to those of DGFF in dimension $d - 2$

Basic properties

- Under P_N, the field is $\begin{cases} \text{delocalized if } d \leq 4 \\ \text{localized if } d \geq 5 \end{cases}$

- $d \geq 5 \Rightarrow \exists P_\infty$: infinite-volume Gibbs measure $\sim \mathcal{N}(0, (-\Delta)^{-2})$

$$\mathrm{Cov}_{P_\infty}(\phi_x, \phi_y) = (-\Delta)^{-2}(x, y) \sim \frac{C}{|x-y|^{d-4}} \text{ as } |x-y| \to \infty$$

Q. Behavior of the field (under external potentials) ?

Let $d = 1$.

Set $\eta_x := \nabla\phi_x = \phi_x - \phi_{x-1}$

$\zeta_x := \Delta\phi_x = \phi_{x+1} - 2\phi_x + \phi_{x-1} = \nabla\eta_{x+1}$

Formally speaking,

under P_N $\left(= \frac{1}{Z_N} \prod_{x\in\mathbb{Z}} e^{-V(\Delta\phi_x)} \prod_{x\in\Lambda_N} d\phi_x \prod_{x\notin\Lambda_N} \delta_0(d\phi_x)\right)$

- $\{\zeta_x\}$: i. i .d.
- $\eta_y = \sum_{z=-N}^{y-1} \zeta_z, \ \ \phi_x = \sum_{y=-N}^{x} \eta_y = \sum_{y=-N}^{x} \left(\sum_{z=-N}^{y-1} \zeta_z \right)$

$\Rightarrow$ P_N $\leftrightarrow$ the law of a (pinned) integrated RW

The One-dimensional case

Precise formulation is as follows :

- $\{X_n\}_{n\geq 1}$: i.i.d. $\mathbb{R}$-valued random variables
$$P(X_1 \in dx) = \frac{1}{Z} e^{-V(x)} dx$$
- $Z_n := \sum_{i=1}^{n} \big(\sum_{j=1}^{i} X_j \big) = \sum_{i=1}^{n} (n-i+1) X_i \ \sim \mu$

Then,

$$P^{\Delta}_{\{1,\cdots,N-1\}}(\,\cdot\,) \stackrel{d}{=} \text{the law of } (Z_1, Z_2, \cdots, Z_{N-1}) \\ \text{under } \mu(\,\cdot\,|Z_N = 0, Z_{N+1} = 0)$$

Localization by weak pinning potentials

Q. Behavior of the field (under external potentials) ?

For a self-potential $\boldsymbol{U} : \mathbb{R} \to \mathbb{R}$, the corresponding model is generally defined as follows :

$$\boldsymbol{P}_{\boldsymbol{\Lambda}}^{\boldsymbol{U}}(d\phi) = \frac{1}{Z_{\boldsymbol{\Lambda}}^{\boldsymbol{U}}} \exp\{-H_{\boldsymbol{\Lambda}}^{\boldsymbol{\Delta}}(\phi) - \sum_{x\in\boldsymbol{\Lambda}} U(\phi_x)\} \prod_{x\in\boldsymbol{\Lambda}} d\phi_x$$

where

$$Z_{\boldsymbol{\Lambda}}^{\boldsymbol{U}} = \int_{\mathbb{R}^{\boldsymbol{\Lambda}}} \exp\{-H_{\boldsymbol{\Lambda}}^{\boldsymbol{\Delta}}(\phi) - \sum_{x\in\boldsymbol{\Lambda}} U(\phi_x)\} \prod_{x\in\boldsymbol{\Lambda}} d\phi_x$$

Square-well pinning

Add weak self-potentials which attract the field to the height level $\mathbf{0}$

Consider a self-potential : $U_1(r) = -bI(|r| \leq a)$, $a \geq 0$, $b \geq 0$

We denote $P_\Lambda^{a,b} := P_\Lambda^{U_1}$, $Z_\Lambda^{a,b} := Z_\Lambda^{U_1}$

We also define the model with δ-pinning by the following :

$$\widetilde{P}_\Lambda^\varepsilon(d\phi) = \frac{1}{\widetilde{Z}_\Lambda^\varepsilon} \exp\{-H_\Lambda^\Delta(\phi)\} \prod_{x\in\Lambda} (\varepsilon\delta_0(d\phi_x) + d\phi_x)$$

where

$$\widetilde{Z}_\Lambda^\varepsilon = \int_{\mathbb{R}^\Lambda} \exp\{-H_\Lambda^\Delta(\phi)\} \prod_{x\in\Lambda} (\varepsilon\delta_0(d\phi_x) + d\phi_x)$$

δ_0 : Dirac mass at $\mathbf{0}$, $\varepsilon > 0$: strength of the pinning

Remark 7

$P_\Lambda^{a,b} \Rightarrow \widetilde{P}_\Lambda^\varepsilon$ as $a \to 0$ and $b \to \infty$ while keeping $2a(e^b - 1) = \varepsilon$

Localization by weak pinning potentials

Theorem 3 (Caravenna-Deuschel (2008), S. (2018))

For every $a \geq 0$, $b \geq 0$ and $\varepsilon \geq 0$ the free energies

$$F(a,b) := \lim_{N\to\infty} \frac{1}{|\Lambda_N|} \log \frac{Z_N^{a,b}}{Z_N}, \quad \widetilde{F}(\varepsilon) := \lim_{N\to\infty} \frac{1}{|\Lambda_N|} \log \frac{\widetilde{Z}_N^{\varepsilon}}{Z_N}$$

exist and the following hold.

- *When* $d = 1$,
  $\forall a > 0\ \exists b_c = b_c(a) > 0$ *s.t.* $F(a,b) > 0$, $\forall b > b_c$ *and*
  $F(a,b) = 0$, $0 \leq \forall b \leq b_c$
  $\exists \varepsilon_c > 0$ *s.t.* $\widetilde{F}(\varepsilon) > 0$, $\forall \varepsilon > \varepsilon_c$ *and* $\widetilde{F}(\varepsilon) = 0$, $0 \leq \forall \varepsilon \leq \varepsilon_c$
- *When* $d \geq 2$,
  $a > 0,\ b > 0 \Rightarrow F(a,b) > 0, \ \ \varepsilon > 0 \Rightarrow \widetilde{F}(\varepsilon) > 0$

Localization by weak pinning potentials

For GMM with weak pinning potentials,

- $d = 1 \Rightarrow \exists$ localization/delocalization transition
- $d \geq 2 \Rightarrow$ Always localized

Remark 8 (DGFF)

$$\lim_{N\to\infty} \frac{1}{|\Lambda_N|} \log \frac{Z_N^{\nabla,a,b}}{Z_N^{\nabla}} > 0 \text{ and } \lim_{N\to\infty} \frac{1}{|\Lambda_N|} \log \frac{\widetilde{Z}_N^{\nabla,\varepsilon}}{Z_N^{\nabla}} > 0$$

for every $d \geq 1$, $\varepsilon > 0$, $a > 0$, $b > 0$

Localization by weak pinning potentials

Next, define

$$\rho_N(a,b) := \frac{1}{|\Lambda_N|} E^{P_N^{a,b}}\left[|\{x \in \Lambda_N; |\phi_x| \le a\}|\right]$$

$$\widetilde{\rho}_N(\varepsilon) := \frac{1}{|\Lambda_N|} E^{\widetilde{P}_N^{\varepsilon}}\left[|\{x \in \Lambda_N; \phi_x = 0\}|\right]$$

Corollary 1

- When $d = 1$, for every $a > 0$ there exists $b_c = b_c(a) > 0$ s.t.
  $\liminf_{N\to\infty} \rho_N(a,b) > 0$ for every $b > b_c$
  $\lim_{N\to\infty} \rho_N(a,b) = 0$ for every $0 \le b < b_c$
- When $d \ge 2$, for every $a > 0$, $b > 0$ it holds that
  $\liminf_{N\to\infty} \rho_N(a,b) > 0$

Similar results hold for $\rho_N(\varepsilon)$.

Localization by weak pinning potentials

$$\because \quad \frac{1}{|\Lambda_N|} \log \frac{Z_N^{a,b}}{Z_N} = \frac{1}{|\Lambda_N|} \int_0^b \frac{\partial}{\partial b'} \log Z_N^{a,b'} db' = \int_0^b \rho_N(a,b') db'$$

$+$ monotonicity of ρ_N $+$ Theorem 3

Localization by weak pinning potentials

Square-well potential + repulsive potential

Consider a self-potential :

$U_2(r) = -bI(0 \le r \le a) + lI(r < 0)$, $a \ge 0$, $b \ge 0$, $l \ge 0$

We denote $P_\Lambda^{a,b,l} := P_\Lambda^{U_2}$, $Z_\Lambda^{a,b,l} := Z_\Lambda^{U_2}$

Theorem 4 (S. (2018))

For every $a \ge 0$, $b \ge 0$ and $l \ge 0$ the free energy

$$F(a,b,l) := \lim_{N\to\infty} \frac{1}{|\Lambda_N|} \log \frac{Z_N^{a,b,l}}{Z_N}$$

exists and when $d \ge 5$, if $a > 0$ and $b > 0$ then $F(a,b,l) > 0$ for every $l \ge 0$.

- $d \ge 5 \Rightarrow$ wetting transition does not occur for GMM

Localization by weak pinning potentials

The case of $\Delta + \nabla$ interactions

Let $\kappa_1 > 0$ and $\kappa_2 > 0$ be fixed and consider a Hamiltonian

$$\mathcal{H}_\Lambda(\phi) = \kappa_1 H_\Lambda^\nabla(\phi) + \kappa_2 H_\Lambda^\Delta(\phi)$$

For $U : \mathbb{R} \to \mathbb{R}$ and $\varepsilon \geq 0$, define

$$\mathcal{Z}_N^U = \int_{\mathbb{R}^{\Lambda_N}} \exp\{-\mathcal{H}_N(\phi) - \sum_{x \in \Lambda_N} U(\phi_x)\} \prod_{x \in \Lambda_N} d\phi_x$$

$$\widetilde{\mathcal{Z}}_N^\varepsilon = \int_{\mathbb{R}^{\Lambda_N}} \exp\{-\mathcal{H}_N(\phi)\} \prod_{x \in \Lambda_N} (\varepsilon \delta_0(d\phi_x) + d\phi_x)$$

We also define $\mathcal{Z}_N := \mathcal{Z}_N^0 = \widetilde{\mathcal{Z}}_N^0$

Localization by weak pinning potentials

$$U_1(r) = -bI(|r| \le a),\ U_2(r) = -bI(0 \le r \le a) + lI(r < 0)$$

Theorem 5 (Borecki-Caravenna (2010), S. (2018))

The free energies

$$\mathcal{F}(a,b) := \lim_{N\to\infty} \frac{1}{|\Lambda_N|} \log \frac{\mathcal{Z}_N^{U_1}}{\mathcal{Z}_N},\ \widetilde{\mathcal{F}}(\varepsilon) := \lim_{N\to\infty} \frac{1}{|\Lambda_N|} \log \frac{\widetilde{\mathcal{Z}}_N^{\varepsilon}}{\mathcal{Z}_N}$$

$$\mathcal{F}(a,b,l) := \lim_{N\to\infty} \frac{1}{|\Lambda_N|} \log \frac{\mathcal{Z}_N^{U_2}}{\mathcal{Z}_N}$$

exist and the following hold.

1. *When* $d \ge 1$, $\mathcal{F}(a,b) > 0$ *for every* $a > 0$, $b > 0$ *and* $\widetilde{\mathcal{F}}(\varepsilon) > 0$ *for every* $\varepsilon > 0$
2. *When* $d \ge 3$, *if* $a > 0$ *and* $b > 0$ *then* $\mathcal{F}(a,b,l) > 0$ *for every* $l \ge 0$

This behavior is the same as the case of DGFF

Existence of the free energy

At first, we have the following :

Lemma 6

Assume that the function $U : \mathbb{R} \to \mathbb{R}$ is bounded from above.
Then, $\exists C_0 = C_0(d) > 0$ s.t. $\forall x_0 \in \Lambda \Subset \mathbb{Z}^d$, we have

$$\int_{\mathbb{R}} e^{-H_\Lambda^\Delta(\phi) - \sum_{x\in\Lambda} U(\phi_x)} d\phi_{x_0} \geq C_0 e^{-H_{\Lambda\setminus\{x_0\}}^\Delta(\phi) - \sum_{x\in\Lambda\setminus\{x_0\}} U(\phi_x)}$$

Existence of the free energy

Divide Λ_N into $K := \left[\frac{2N+1}{2n+3}\right]^d$ disjoint small boxes with side-length $2n+1$ by imposing 0-boundary conditions with width 2.

Then, by Lemma 6 and the Markov property of the field, we have

$$Z_N^U \geq C_0^{(CKn^{d-1}+C'nN^{d-1})}(Z_n^U)^K$$

By taking $\liminf_{N\to\infty} \frac{1}{|\Lambda_N|} \log \ \cdot$,

$$\liminf_{N\to\infty} \frac{1}{|\Lambda_N|} \log Z_N^U \geq \frac{C}{n} + \left(\frac{2n+1}{2n+3}\right)^d \frac{1}{|\Lambda_n|} \log Z_n^U$$

Finally, by taking $\limsup_{n\to\infty}$ we obtain that

$$\liminf_{N\to\infty} \frac{1}{|\Lambda_N|} \log Z_N^U \geq \limsup_{n\to\infty} \frac{1}{|\Lambda_n|} \log Z_n^U$$

Positivity of $F(a,b)$

Key idea

- Add appropriate mass terms
- Comparison between $(\Delta_N^2)^{-1}$ and $(-\Delta_N)^{-2}$

For $m \geq 0$, consider a Hamiltonian

$$H_{N,m}(\phi) = H_N^{\Delta}(\phi) + 2m^2 H_N^{\nabla}(\phi) + \frac{1}{2} m^4 \sum_{x \in \Lambda_N} (\phi_x)^2$$

For a self-potential $U : \mathbb{R} \to \mathbb{R}$, define

$$P_{N,m}^U(d\phi) = \frac{1}{Z_{N,m}^U} \exp\{-H_{N,m}(\phi) - \sum_{x \in \Lambda_N} U(\phi_x)\} \prod_{x \in \Lambda_N} d\phi_x$$

$$Z_{N,m}^U = \int_{\mathbb{R}^{\Lambda_N}} \exp\{-H_{N,m}(\phi) - \sum_{x \in \Lambda_N} U(\phi_x)\} \prod_{x \in \Lambda_N} d\phi_x$$

Positivity of $F(a,b)$

By the definition, $Z^U_{N,m} \le Z^U_N$ and this yields that

$$
\begin{aligned}
F(a,b) &:= \lim_{N\to\infty} \frac{1}{|\Lambda_N|} \log \frac{Z_N^{U_1}}{Z_N} \\
&\ge \liminf_{N\to\infty} \frac{1}{|\Lambda_N|} \log \frac{Z_{N,m}^{U_1}}{Z_N} \\
&\ge \liminf_{N\to\infty} \frac{1}{|\Lambda_N|} \log \frac{Z_{N,m}^{U_1}}{Z_{N,m}} + \liminf_{N\to\infty} \frac{1}{|\Lambda_N|} \log \frac{Z_{N,m}}{Z_N} \\
&=: I_1 + I_2
\end{aligned}
$$

$I_1 \leftrightarrow$ free energy of the massive model with pinning

$I_2 \leftrightarrow$ cost of adding mass terms

Positivity of $F(a,b)$

Estimate on I_1 :

First of all, by Jensen's inequality

$$\begin{aligned}\log \frac{Z_{N,m}^{U_1}}{Z_{N,m}} &= \log E^{P_{N,m}}\Big[\exp\{-\sum_{x\in\Lambda_N} U_1(\phi_x)\}\Big] \\ &\geq b \sum_{x\in\Lambda_N} P_{N,m}(|\phi_x| \leq a)\end{aligned} \tag{1}$$

Positivity of $F(a,b)$

The following deformation of $H_N^{\Delta}(\phi)$ is helpful

$$
\begin{aligned}
H_N^{\Delta}(\phi) &= \frac{1}{2}\sum_{x\in\overline{\Lambda}_N}(\Delta\phi_x)^2\Big|_{\phi\equiv 0 \text{ on } \partial_2^+\Lambda_N} \\
&= \frac{1}{2}\sum_{x\in\Lambda_N}\Big(\sum_{y\in\Lambda_N}\Delta(x,y)\phi_y\Big)\Big(\sum_{z\in\Lambda_N}\Delta(x,z)\phi_z\Big) \\
&\qquad +\frac{1}{2}\sum_{x\in\partial^+\Lambda_N}\Big(\sum_{y\in\Lambda_N}\Delta(x,y)\phi_y\Big)\Big(\sum_{z\in\Lambda_N}\Delta(x,z)\phi_z\Big) \\
&= \frac{1}{2}\langle\phi,\,(-\Delta_N)^2\phi\rangle_N + B_N(\phi)
\end{aligned}
$$

where

$$
B_N(\phi)=\sum_{x\in\partial^-\Lambda_N}\frac{r_N(x)}{8d^2}(\phi_x)^2,\; r_N(x)=|\{y\in\partial^+\Lambda_N;|y-x|=1\}|
$$

Positivity of $F(a, b)$

Therefore,

$$\begin{aligned} H_{N,m}(\phi) &= \frac{1}{2}\langle \phi,\, (-\Delta_N)^2\phi\rangle_N + B_N(\phi) \\ &\qquad + m^2\langle \phi,\, (-\Delta_N)\phi\rangle_N + \frac{1}{2}m^4\langle \phi,\, \phi\rangle_N \\ &= \frac{1}{2}\langle \phi,\, (m^2 I_N - \Delta_N)^2\phi\rangle_N + B_N(\phi) \end{aligned}$$

$\Rightarrow P_{N,m} = \mathcal{N}(0, (m^2 I_N - \Delta_N)^{-2})$

+ convex self-potentials on $\partial^- \Lambda_N$

Positivity of $F(a, b)$

Let

$Q_{N,m} \sim \mathcal{N}(0, (m^2 I_N - \Delta_N)^{-2})$
$Q_{\infty,m} \sim \mathcal{N}(0, (m^2 I - \Delta)^{-2})$

and define

$$\Xi_{N,m} = \int_{\mathbb{R}^{\Lambda_N}} \exp\Big\{-\frac{1}{2}\langle \phi,\ (m^2 I_N - \Delta_N)^2 \phi\rangle_N\Big\} \prod_{x \in \Lambda_N} d\phi_x$$

Remark 9

$m > 0 \Rightarrow Q_{\infty,m}$ exists for every $d \geq 1$. $Q_{\infty,0}$ exists for $d \geq 5$

Positivity of $F(a,b)$

Then, we have

$$\mathbf{Var}_{P_{N,m}}(\phi_x) \le \mathbf{Var}_{Q_{N,m}}(\phi_x) \le \mathbf{Var}_{Q_{\infty,m}}(\phi_0)$$

for every $x \in \Lambda_N$

$\because$ Brascamp-Lieb inequality

$$\mu(d\phi) = \frac{1}{Z}\exp\Big\{-\frac{1}{2}\langle \phi, A\phi\rangle_{\Gamma}\Big\}\prod_{x\in\Gamma} d\phi_x$$

: Gaussian measure on $\mathbb{R}^{\Gamma}$ ($|\Gamma| < \infty$)

$$\mu^U(d\phi) \propto \exp\Big\{-\sum_{x\in\Gamma} U_x(\phi_x)\Big\}\mu(d\phi)$$

$U_x : \mathbb{R} \to \mathbb{R}$: convex

$$\Rightarrow \mathbf{Var}_{\mu^U}\big(\langle \nu, \phi\rangle_{\Gamma}\big) \le \mathbf{Var}_{\mu}\big(\langle \nu, \phi\rangle_{\Gamma}\big) \text{ for every } \nu \in \mathbb{R}^{\Gamma}$$

Positivity of $F(a,b)$

$P(|X| \leq a) \downarrow$ as $\sigma^2 \uparrow$ for a Gaussian r.v. $X \sim \mathcal{N}(0,\sigma^2)$ and $a > 0$

By (1) and these estimates, we obtain the following :

$$I_1 := \liminf_{N\to\infty} \frac{1}{|\Lambda_N|} \log \frac{Z_{N,m}^{U_1}}{Z_{N,m}} \geq b\, Q_{\infty,m}(|\phi_0| \leq a)$$

Estimate on I_2 :

By the previous observation we can show the following :

$$\liminf_{N\to\infty} \frac{1}{|\Lambda_N|} \log \frac{Z_{N,m}}{Z_N} \geq \liminf_{N\to\infty} \frac{1}{|\Lambda_N|} \log \frac{\Xi_{N,m}}{\Xi_{N,0}} \qquad (2)$$

Positivity of $F(a, b)$

By the definition,

$$\begin{aligned}\Xi_{N,m} &= (2\pi)^{\frac{|\Lambda_N|}{2}} \sqrt{\det((m^2 I_N - \Delta_N)^{-2})} \\ &= (2\pi)^{\frac{|\Lambda_N|}{2}} \det\big((m^2 I_N - \Delta_N)^{-1}\big)\end{aligned}$$

RW representation for $(m^2 I_N - \Delta_N)^{-1}$ yields that

$$\begin{aligned}\log \Xi_{N,m} = \frac{1}{2}|\Lambda_N| \log(2\pi) - |\Lambda_N| \log(m^2 + 1) \\ + \sum_{x \in \Lambda_N} \sum_{n=1}^{\infty} \frac{1}{n} \Big(\frac{1}{m^2 + 1}\Big)^n \mathbb{P}_x(S_n = x, n < \tau_{\Lambda_N})\end{aligned}$$

Positivity of $F(a, b)$

By this representation we can show the following asymptotics.

Lemma 2

The following hold as $m \downarrow 0$.

$$(a) \quad \limsup_{N\to\infty}\Big\{-\frac{1}{|\Lambda_N|}\log\frac{\Xi_{N,m}}{\Xi_{N,0}}\Big\}$$

$$\leq J_d(m) := \begin{cases} C_d m(1+o(1)) & \text{if } d=1 \\ C_d m^2|\log m|(1+o(1)) & \text{if } d=2 \\ C_d m^2(1+o(1)) & \text{if } d\geq 3 \end{cases}$$

$$(b) \quad \max_{x\in\Lambda_N} \mathrm{Var}_{Q_{N,m}}(\phi_x) \leq \mathrm{Var}_{Q_{\infty,m}}(\phi_0)$$

$$= \sigma_d^2(m) := \begin{cases} \widetilde{C}_d m^{-4+d}(1+o(1)) & \text{if } d=1,2,3 \\ \widetilde{C}_d|\log m|(1+o(1)) & \text{if } d=4 \\ \widetilde{C}_d(1+o(1)) & \text{if } d\geq 5 \end{cases}$$

Positivity of $F(a,b)$

(2) and Lemma 2 yield that

$$I_2 := \liminf_{N\to\infty} \frac{1}{|\Lambda_N|} \log \frac{Z_{N,m}}{Z_N} \ge -J_d(m)$$

Collecting all the estimates, for every $m > 0$ we have

$$\begin{aligned} F(a,b) &:= \lim_{N\to\infty} \frac{1}{|\Lambda_N|} \log \frac{Z_N^{U_1}}{Z_N} \\ &\ge b\, Q_{\infty,m}(|\phi_0| \le a) - J_d(m) \\ &\ge \frac{2ab}{\sqrt{2\pi\sigma_d^2(m)}} e^{-\frac{a^2}{2\sigma_d^2(m)}} - J_d(m) \end{aligned}$$

Finally, by Lemma 2 $\frac{1}{\sqrt{\sigma_d^2(m)}} \gg J_d(m)$ as $m \downarrow 0$ when $d \ge 2$.

By taking $m > 0$ small enough, we obtain $F(a,b) > 0$ for every $a > 0$, $b > 0$ when $d \ge 2$.

□

The case of $\Delta + \nabla$ interactions

By simple change of variables we may assume that $\kappa_1 = \kappa > 0$, $\kappa_2 = 1$

$$
\begin{aligned}
\mathcal{H}_N(\phi) &:= \kappa H_N^{\nabla}(\phi) + H_N^{\Delta}(\phi) \\
&= \frac{1}{2}\kappa \langle \phi, (-\Delta_N)\phi \rangle_N + \frac{1}{2}\langle \phi, (-\Delta_N)^2 \phi \rangle_N + B_N(\phi) \\
&= \frac{1}{2}\langle \phi, (-\Delta_N)(\kappa I_N - \Delta_N)\phi \rangle_N + B_N(\phi)
\end{aligned}
$$

Behavior of $(-\Delta_N)^{-1}(\kappa I - \Delta_N)^{-1}$ is similar to that of $(-\Delta_N)^{-1}$

Exponential decay of correlations by pinning effects

Q. Pathwise description of the localization by pinning effects ?

Theorem 7 (Bolthausen-Cipriani-Kurt (2017))

Let $d \geq 5$ and $\varepsilon > 0$.
There exist $C_1 = C_1(d, \varepsilon) > 0$, $C_2 = C_2(d, \varepsilon) > 0$ s.t.

$$\left| E^{\widetilde{P}_N^{\varepsilon}}[\phi_x \phi_y] \right| \leq C_1 e^{-C_2 |x-y|}$$

for every $x, y \in \mathbb{Z}^d$.

Exponential decay of correlations by pinning effects

Idea for the proof of Theorem 7

For $f : \mathbb{R}^{\mathbb{Z}^d} \to \mathbb{R}$, we have

$$\begin{aligned}
E^{\widetilde{P}_N^\varepsilon}[f] &= \frac{1}{\widetilde{Z}_N^\varepsilon} \int f(\phi) \exp\{-H_N(\phi)\} \\
&\qquad \times \prod_{x \in \Lambda_N} (\varepsilon \delta_0(d\phi_x) + d\phi_x) \prod_{x \notin \Lambda_N} \delta_0(d\phi_x) \\
&= \sum_{A \subset \Lambda_N} \varepsilon^{|A|} \frac{1}{\widetilde{Z}_N^\varepsilon} \int f(\phi) \exp\{-H_N(\phi)\} \\
&\qquad \times \prod_{x \in \Lambda_N \setminus A} d\phi_x \prod_{x \notin A} \delta_0(d\phi_x) \\
&= \sum_{A \subset \Lambda_N} \varepsilon^{|A|} \frac{Z_{\Lambda_N \setminus A}}{\widetilde{Z}_N^\varepsilon} E^{P_{\Lambda_N \setminus A}}[f]
\end{aligned}$$

Exponential decay of correlations by pinning effects

Define $\mathcal{A} = \mathcal{A}_N(\phi) := \{x \in \Lambda_N; \phi_x = 0\}$ (random pinned region)
Then, we have

$$\widetilde{P}_N^{\varepsilon}(\mathcal{A} = A) = \varepsilon^{|A|} \frac{Z_{\Lambda_N \setminus A}}{\widetilde{Z}_N^{\varepsilon}} =: \zeta_N^{\varepsilon}(A)$$

Therefore,

$$E^{\widetilde{P}_N^{\varepsilon}}[\phi_x \phi_y] = \sum_{A \subset \Lambda_N} \zeta_N^{\varepsilon}(A) E^{P_{\Lambda_N \setminus A}}[\phi_x \phi_y] \tag{3}$$

Remark 10

This expansion holds for both of DGFF and GMM

The case of DGFF (Bolthausen-Velenik (2001))

By random walk representation of the covariance

$$E^{P^{\nabla}_{\Lambda_N\setminus A}}[\phi_x\phi_y] = (-\Delta_{\Lambda_N\setminus A})^{-1}(x,y)$$
$$= \mathbb{E}_x\Big[\sum_{n\geq 0} I(S_n = y, n < \tau_{\Lambda_N\setminus A})\Big]$$

Substitute this for (3),

$$E^{\widetilde{P}^{\varepsilon,\nabla}_N}[\phi_x\phi_y] = \sum_{n\geq 0}\mathbb{E}_x\Big[I(S_n = y, n < \tau_{\Lambda_N})\zeta^{\varepsilon}_N(\mathcal{A}\cap S_{[0,n]} = \phi)\Big]$$

where $S_{[0,n]} := \{S_k; 0 \leq k \leq n\}$

Under $\widetilde{P}_N^{\varepsilon}$, $\{\zeta_N^{\varepsilon}(A); A \subset \Lambda_N\}$ can be compared with Bernoulli measure ν_ρ on $\{0,1\}^{\mathbb{Z}^d}$

For every $\varepsilon > 0$ there exist $\rho_\pm = \rho_\pm(d,\varepsilon) \in (0,1)$ such that

$$\nu_{\rho_-}(\mathcal{A} \cap B = \phi) \le \zeta_N^{\varepsilon}(\mathcal{A} \cap B = \phi) \le \nu_{\rho_+}(\mathcal{A} \cap B = \phi)$$

for every $B \subset \Lambda_N$

Hence we obtain

$$E^{\widetilde{P}_N^{\varepsilon,\nabla}}[\phi_x \phi_y] \le \sum_{n \ge 0} \mathbb{E}_x\Big[I(S_n = y)(1-\rho_+)^{|S_{[0,n]}|}\Big]$$

Exponential decay of correlations by pinning effects

The case of GMM (Bolthausen et al. (2017))

RW representation of the covariance :
$E^{P^{\Delta}_{\Lambda_N \setminus A}}[\phi_x \phi_y] = (\Delta^2_{\Lambda_N \setminus A})^{-1}(x, y)$
does not hold

For $x \in \Lambda_N$, $G_A^N(x, \cdot) := (\Delta^2_{\Lambda_N \setminus A})^{-1}(x, \cdot)$ satisfies the following Dirichlet problem for bi-Laplacian

$$\begin{cases} \Delta^2 G_A^N(x, y) = \delta(x, y) & \text{for } \; y \in \Lambda_N \setminus A \\ G_A^N(x, y) = 0 & \text{for } \; y \in A \cup \Lambda_N^c \end{cases} \tag{4}$$

Exponential decay of correlations by pinning effects

By stochastic domination for ζ_N^ε
+ analytic estimates for the solution of (4) (with random Dirichlet b.c.)
+ percolation argument,
the following holds.

Proposition 4

Let $d \geq 5$ and $\varepsilon > 0$.
There exist $K = K(d, \varepsilon) > 0$, $\delta = \delta(d, \varepsilon) > 0$, $C = C(d, \varepsilon) > 0$ s.t. the following holds for every $n \geq 1$

$$\sup_{N \geq 1} \zeta_N^\varepsilon \Big(\max_{Kn \leq |x| \leq K(n+1)} |G_{\mathcal{A}}^N(0, x)| \geq e^{-\delta n} \Big) \leq C e^{-\delta n}$$

Exponential decay of correlations by pinning effects

Then, for $x \in \mathbb{Z}^d : Kn \le |x| \le K(n+1)$, we have

$$\begin{aligned}
\sup_{N\ge 1}\left|E^{\widetilde{P}_N^\varepsilon}\left[\phi_0\phi_x\right]\right| &= \sup_{N\ge 1}\left|E^{\zeta_N^\varepsilon}\left[G_{\mathcal{A}}^N(0,x)\right]\right| \\
&\le \sup_{N\ge 1} E^{\zeta_N^\varepsilon}\left[\left|G_{\mathcal{A}}^N(0,x)\right| I(0\notin\mathcal{A})\ ; F_n\right] \\
&\quad + \sup_{N\ge 1} E^{\zeta_N^\varepsilon}\left[\left|G_{\mathcal{A}}^N(0,x)\right| I(0\notin\mathcal{A})\ ; F_n^c\right] \\
&\le GCe^{-\delta'|x|} + e^{-\delta'|x|}
\end{aligned}$$

where F_n represents the event in Proposition 4 and $G = (-\Delta)^{-2}(0,0)$

Entropic repulsion

Q. Behavior of the field interacting with a hard wall ?

$\Omega^+(A) := \{\phi\,;\, \phi_x \geq 0 \ \forall x \in A\}$, $A \subset \mathbb{Z}^d$

$D \subset \Lambda := [-1, 1]^d$ domain with piecewise smooth boundary

$\mathrm{dist}(\partial D, \Lambda^c) \geq \delta > 0$

$D_N := N\overline{D} \cap \mathbb{Z}^d$

$\Lambda_L(x) := x + \Lambda_L$, $x \in \mathbb{Z}^d$, $L \geq 1$

Entropic repulsion

Theorem 8 (Kurt '07, '09, S. '03, '16)

- *When* $d \geq 5$,

$$\lim_{N\to\infty} \frac{1}{N^{d-4}\log N} \log P_N^{\Delta}(\Omega^+(D_N)) = -4(-\Delta)^{-2}(0,0)\mathrm{Cap}_{\Lambda}^{\Delta}(D)$$

- *When* $d = 4$,

$$\lim_{N\to\infty} \frac{1}{(\log N)^2} \log P_N^{\Delta}(\Omega^+(D_N)) = -\frac{64}{\pi^2}\mathrm{Cap}_{\Lambda}^{\Delta}(D)$$

- *When* $d = 1, 2, 3$,
  $\forall \delta \in (0,1)$, $\exists \gamma > 0$ *(small) and* $\exists C > 0$
  s.t. $\forall N \geq 1$, $\forall x \in \Lambda_{\delta N}$, *it holds that*

$$P_N^{\Delta}(\phi_y \geq 0 \text{ for every } y \in \Lambda_{\gamma N}(x)) \geq C$$

Entropic repulsion

Remark 11 (DGFF)

- When $d \geq 3$,

$$\lim_{N\to\infty} \frac{1}{N^{d-2}\log N} \log P_N^{\nabla}\big(\Omega^+(D_N)\big) = -2(-\Delta)^{-1}(0,0)\mathrm{Cap}_{\Lambda}^{\nabla}(D)$$

- When $d = 2$,

$$\lim_{N\to\infty} \frac{1}{(\log N)^2} \log P_N^{\nabla}\big(\Omega^+(D_N)\big) = -\frac{4}{\pi}\mathrm{Cap}_{\Lambda}^{\nabla}(D)$$

- When $d = 1$,
  $\forall \delta \in (0,1)$, $\exists C > 0$ s.t. $\forall N \geq 1$, it holds that

$$P_N^{\nabla}\big(\phi_x \geq 0 \text{ for every } x \in \Lambda_{\delta N}\big) \geq C$$

Entropic repulsion

Theorem 9 (Kurt '07, '09, S. '03)

Set $\overline{\phi}_{\varepsilon N}(x) := \frac{1}{|\Lambda_{\varepsilon N}(x)|} \sum_{y \in \Lambda_{\varepsilon N}(x)} \phi_y$

For every $0 < \varepsilon < 1, \delta > 0$*, the following hold.*

- *When* $d \geq 5$*,*

$$\lim_{N \to \infty} \min_{x : \Lambda_{\varepsilon N}(x) \subset D_N} P_N\Big(\Big| \frac{\overline{\phi}_{\varepsilon N}(x)}{\sqrt{\log N}} - \sqrt{8G} \Big| \leq \delta \;\Big|\; \Omega^+(D_N) \Big) = 1$$

  where $G = (-\Delta)^{-2}(0,0)$

- *When* $d = 4$*,*

$$\lim_{N \to \infty} \min_{x : \Lambda_{\varepsilon N}(x) \subset D_N} P_N\Big(\Big| \frac{\overline{\phi}_{\varepsilon N}(x)}{\log N} - \sqrt{8\gamma} \Big| \leq \delta \;\Big|\; \Omega^+(D_N) \Big) = 1$$

  where $\gamma = \frac{8}{\pi^2}$

Entropic repulsion

Remark 12

$$E^{P_N}\Big[\max_{x\in\Lambda_N}\phi_x\Big] = \begin{cases} O(\sqrt{\log N}) & \text{when } d \geq 5 \\ O(\log N) & \text{when } d = 4 \end{cases}$$

$$\mathrm{Var}_{P_N}(\phi_0) = \begin{cases} O(1) & \text{when } d \geq 5 \\ O(\log N) & \text{when } d = 4 \end{cases}$$

When $d \geq 4$, the field is pushed up to the same level as $E^{P_N}[\max_{x\in\Lambda_N}\phi_x]$ by the hard wall condition $\Omega^+(\Lambda_N)$

$\leftrightarrow$ Entropic repulsion

Remark 13

When $d = 1, 2, 3$

$E^{P_N}\Big[\max_{x\in\Lambda_N}\phi_x\Big] = O(N^{2-\frac{d}{2}})$, $\mathrm{Var}_{P_N}(\phi_0) = O(N^{4-d})$

$\Rightarrow$ Entropic repulsion does not occur

Proof of $P_N\big(\Omega^+(D_N)\big)$ upper bound

Idea for the proof of $P_N\big(\Omega^+(D_N)\big)$ upper bound $(d \geq 5)$

Let $P_N \sim \mathcal{N}(0, (\Delta_N^2)^{-1})$, $d \geq 5$

$\Lambda_N := [-N, N]^d \cap \mathbb{Z}^d$, $D \subset [-1, 1]^d$, $D_N := N\overline{D} \cap \mathbb{Z}^d$

For $L \geq 1$, define

$$\widetilde{\Lambda}_N = \widetilde{\Lambda}_{N,L} := \{x \in 4L\mathbb{Z}^d ; \partial_+^2 \Lambda_L(x) \subset D_N\}$$

$$\Gamma_N = \Gamma_{N,L} := \bigcup_{x \in \widetilde{\Lambda}_N} \partial_+^2 \Lambda_L(x)$$

$$m_x^L(\phi) := E^{P_N}\big[\, \phi_x \,\big|\, \mathcal{F}_{\Gamma_N} \,\big](\phi)$$

Then, by the Markov property of the field we have the following :

Under $P_N\big(\,\cdot\, | \mathcal{F}_{\Gamma_N}\big)(\phi)$,

$\{\phi_x ; x \in \widetilde{\Lambda}_N\}$ are independent and

$\phi_x \sim \mathcal{N}\big(m_x^L(\phi),\ G_L\big)$ where $G_L = (\Delta_L^2)^{-1}(0,0)$

Proof of $P_N(\Omega^+(D_N))$ upper bound

Now, consider an event : $\mathcal{E} = \{|\mathcal{A}| \geq \delta|\widetilde{\Lambda}_N|\}$ $\quad (\varepsilon > 0, \delta > 0)$

where $\mathcal{A} = \mathcal{A}_N(\phi) := \{x \in \widetilde{\Lambda}_N; m_x^L(\phi) \leq \sqrt{(8G - \varepsilon)\log N}\}$

For $x \in \mathcal{A}$, if $\phi_x \geq 0$ then $\phi_x - m_x^L(\phi) \geq -\sqrt{(8G - \varepsilon)\log N}$

Therefore, by the Markov property of the field

$$P_N\big(\Omega^+(D_N) \cap \mathcal{E}\big) \leq E^{P_N}\Big[\prod_{x \in \widetilde{\Lambda}_N} P_N\big(\phi_x \geq 0 \mid \mathcal{F}_{\Gamma_N}\big); \mathcal{E}\Big]$$

$$\leq \Big(1 - \frac{C}{\sqrt{\log N}} \exp\{-\frac{8G - \varepsilon}{2G_L} \log N\}\Big)^{\delta|\widetilde{\Lambda}_N|}$$

Since $G_L \to G$ as $L \to \infty$, by taking L large enough we have

$$P_N\big(\Omega^+(D_N) \cap \mathcal{E}\big) \leq \exp\{-C' N^{d-4G+\delta'}\}$$

for some $\delta' > 0$, $C' > 0$

Proof of $P_N(\Omega^+(D_N))$ upper bound

Next, on $\mathcal{E}^c$

$$\begin{aligned}\sum_{x\in\widetilde{\Lambda}_N} m_x^L(\phi) &\geq \sum_{x\in\mathcal{A}^c} m_x^L(\phi) + \sum_{x\in\mathcal{A}} m_x^L(\phi)\\ &\geq \sqrt{(8G-\varepsilon)\log N}(1-\delta)|\widetilde{\Lambda}_N| + \sum_{x\in\mathcal{A}} m_x^L(\phi)\end{aligned}$$

Now, assume that

$$\sum_{x\in\mathcal{A}} m_x^L(\phi) \geq 0 \quad \text{on } \Omega^+(D_N) \tag{5}$$

Then, we have

$$\frac{1}{|\widetilde{\Lambda}_N|}\sum_{x\in\widetilde{\Lambda}_N} m_x^L(\phi) \geq (1-\delta)\sqrt{(8G-\varepsilon)\log N} \quad \text{on } \Omega^+(D_N)\cap\mathcal{E}^c$$

Proof of $P_N(\Omega^+(D_N))$ upper bound

By Jensen's inequality,

$$\mathrm{Var}_{P_N}\Big(\frac{1}{|\widetilde{\Lambda}_N|}\sum_{x\in\widetilde{\Lambda}_N} m_x^L(\phi)\Big) \le E^{P_N}\Big[\Big(\frac{1}{|\widetilde{\Lambda}_N|}\sum_{x\in\widetilde{\Lambda}_N}\phi_x\Big)^2\Big] = O(N^{-d+4})$$

Therefore,

$$\begin{aligned}
&P_N\big(\Omega^+(D_N)\cap\mathcal{E}^c\big)\\
&\le P_N\Big(\frac{1}{|\widetilde{\Lambda}_N|}\sum_{x\in\widetilde{\Lambda}_N} m_x^L(\phi) \ge (1-\delta)\sqrt{(8G-\varepsilon)\log N}\Big)\\
&\le \exp\Big\{-\frac{1}{2\mathrm{Var}_{P_N}\Big(\frac{1}{|\widetilde{\Lambda}_N|}\sum_{x\in\widetilde{\Lambda}_N} m_x^L(\phi)\Big)}(1-\delta)^2(8G-\varepsilon)\log N\Big\}\\
&\le e^{-CN^{d-4}\log N}
\end{aligned}$$

Proof of $P_N(\Omega^+(D_N))$ upper bound

Hence, we obtain

$$\limsup_{N\to\infty} \frac{1}{N^{d-4}\log N} \log P_N\big(\Omega^+(D_N)\big) \le -C$$

for some $C > 0$

On the assumption (5)

The case of DGFF

$$\begin{aligned} m_x^L(\phi) &:= E^{P_N^\nabla}\big[\,\phi_x \,\big|\, \mathcal{F}_{\Gamma_N}\big](\phi) \\ &= \sum_{y\in\partial^+\Lambda_L(x)} \phi_y \mathbb{P}_x\big(S(\tau_{\Lambda_L(x)}) = y\big) \\ &\ge 0 \quad \text{for every } x \in \widetilde{\Lambda}_N \text{ on } \Omega^+(D_N) \end{aligned}$$

The case of GMM

Need to show that $\sum_{x\in\mathcal{A}} m_x^L(\phi)$ is negligible on $\Omega^+(D_N) \cap \mathcal{E}^c$