A CENTRAL LIMIT THEOREM FOR THE EFFECTIVE CONDUCTANCE

MAREK BISKUP

As is well known, most metals, regardless how pristine they may seem at the macroscopic scale, have a rather complicated microscopic structure. This naturally leads to the question why their conductivity properties are governed by equations with smooth coefficients. An answer is offered by homogenization theory: smoothness arises via self-averaging.

In my talk I will consider a specific problem of electric conductivity of random resistor networks on $\mathbb{Z}^d$. To each unordered pair $\langle x, y\rangle$ of nearest-neighbor vertices we assign a resistance $r_{xy} = r_{yx}$ or, alternatively, a conductance $c_{xy} = 1/r_{xy}$. For a finite connected set $V \subset \mathbb{Z}^d$, with incident edges, resp., outer boundary vertices denoted by $\mathbb{B}(V)$, resp., ∂V, we are looking for the minimum of the Dirichlet energy

$$Q_V(f) := \sum_{\langle x,y\rangle\in\mathbb{B}(V)} c_{xy}\bigl[f(y) - f(x)\bigr]^2$$

over all $f\colon V \cup \partial V \to \mathbb{R}$ such that $f = g$ on ∂V. In the specific case of a square box $V_L := (0, L)^d \cap \mathbb{Z}^d$ and linearly-growing boundary conditions, $g(x) := t \cdot x$, we define the effective conductance by

$$C_L^{\text{eff}}(t) := \inf\bigl\{Q_{V_L}(f)\colon f(x) = t \cdot x,\ \forall x \in \partial V_L\bigr\}$$

Assuming that the conductances c_{xy} are independent and identically distributed, we aim to describe the asymptotic distributional properties of $C_L^{\text{eff}}(t)$ when L is very large.

It is well known that, under mild integrability conditions on c_{xy}, the quantity $C_L^{\text{eff}}(t)/L^d$ scales to a deterministic constant. Assuming a small ellipticity contrast, i.e., the existence of a small-enough $\lambda > 1$ such that

$$\frac{1}{\lambda} \le c_{xy} \le \lambda,$$

I will outline the proof of a central limit theorem,

$$\frac{C_L^{\text{eff}}(t) - \mathbb{E}C_L^{\text{eff}}(t)}{L^{d/2}} \underset{L\to\infty}{\overset{\text{law}}{\longrightarrow}} \mathcal{N}(0, \sigma_t^2),$$

for some σ_t^2 which is positive whenever $t \neq 0$ and is, in fact, of a bi-quadratic form in t. Among the key ingredients are the corrector method from homogenization theory and the Martingale Central Limit Theorem; an integrability condition is furnished by the Meyers estimate.

The talk is based on a joint paper with M. Salvi and T. Wolff. Time permitting, I will comment on related results due to P. Nolen and, independently, R. Rosignol that deal with periodic boundary conditions, and a method to remove the restriction on the ellipticity contrast.