KPZ equation, its renormalization and invariant measures

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The Kardar-Parisi-Zhang (KPZ) equation is a stochastic partial differential equation of the form

$$\partial_t h = \frac{1}{2}\partial_x^2 h + \frac{1}{2}\big(\partial_x h\big)^2 + \dot{W}(t,x), \quad x \in \mathbb{R},$$

where $\dot{W}(t,x)$ is the space-time Gaussian white noise, which has the correlation function

$$E[\dot{W}(t,x)\dot{W}(s,y)] = \delta(x-y)\delta(t-s).$$

We consider in one dimension on the whole line. This equation is ill-posed because of inconsistency between the nonlinearity and the roughness of the noise. However, its Cole-Hopf solution defined as the logarithm of the solution of the linear stochastic heat equation (SHE) with a multiplicative noise:

$$\partial_t Z = \frac{1}{2}\partial_x^2 Z + Z\dot{W}(t,x), \quad x \in \mathbb{R},$$

i.e., $h(t,x) := \log Z(t,x)$ is a mathematically well-defined object and, in fact, M. Hairer has recently proved that the solution of SHE can be derived through the Cole-Hopf transform of the solution of the KPZ equation with a suitably renormalized factor under the periodic boundary condition.

In this talk, we introduce a different type of renormalization for the KPZ equation on the whole line:

$$\partial_t h = \frac{1}{2}\partial_x^2 h + \frac{1}{2}\big((\partial_x h)^2 - \xi^\varepsilon\big) * \eta_2^\varepsilon + \dot{W}^\varepsilon(t,x), \quad x \in \mathbb{R},$$

where a smeared noise $\dot{W}^\varepsilon(t,x) = \dot{W} * \eta^\varepsilon(t,x)$, $\eta_2^\varepsilon = \eta_\varepsilon * \eta_\varepsilon$ and $\xi^\varepsilon = \eta_2^\varepsilon(0)$ are defined from a usual convolution kernel η^ε which tends to δ_0 as $\varepsilon \downarrow 0$. This type of renormalization is appropriate from the view point to characterize the invariant measures. The Cole-Hopf transform applied to this equation leads to an SHE with a smeared noise having an extra complex nonlinear term involving a certain renormalization structure:

$$\partial_t Z = \frac{1}{2}\partial_x^2 Z + \frac{1}{2} Z \left\{ \left(\frac{\partial_x Z}{Z}\right)^2 * \eta_2^\varepsilon - \left(\frac{\partial_x Z}{Z}\right)^2 \right\} + Z\dot{W}^\varepsilon(t,x), \quad x \in \mathbb{R}.$$

It is shown that, under the situation that the corresponding tilt process is stationary, this complex term (the middle term in the RHS) can be replaced by a simple linear term divided by a specific constant 24 in the limit, so that the limit equation is the linear SHE:

$$\partial_t Z = \frac{1}{2}\partial_x^2 Z + \frac{1}{24} Z + Z\dot{W}(t,x), \quad x \in \mathbb{R}.$$

The Wiener-Itô expansion and a similar method for establishing the so-called Boltzmann-Gibbs principle are effectively used. As a result, it is shown that the distribution of a two-sided geometric Brownian motion with a height shift given by Lebesgue measure is invariant under the evolution determined by the SHE on $\mathbb{R}$.

Multi-component KPZ equation will be also discussed at approximating level and we study its invariant measures.

This is a joint work with Jeremy Quastel.