KPZ equation with fractional derivatives of white noise

Masato Hoshino (The University of Tokyo)

We discuss the stochastic partial differential equation

$$\partial_t h(t,x) = \partial_x^2 h(t,x) + (\partial_x h(t,x))^2 + \partial_x^\gamma \xi(t,x)$$

for $(t,x) \in [0,\infty) \times \mathbb{T}$ with $\gamma \geq 0$. Here, $h(t,x)$ is a continuous stochastic process, and ξ is a space-time white noise on $[0,\infty) \times \mathbb{T}$. Moreover, $\partial_x^\gamma = -(-\partial_x^2)^{\frac{\gamma}{2}}$ is the fractional Laplacian. When $\gamma = 0$, this equation is called the KPZ equation. Recently, M. Hairer discussed the solvability of the KPZ equation. He showed that the renormalized equation

$$\partial_t h_\epsilon(t,x) = \partial_x^2 h_\epsilon(t,x) + (\partial_x h_\epsilon(t,x))^2 - C_\epsilon + \xi_\epsilon(t,x),$$

for a smoothed noise $\xi_\epsilon = \xi * \rho_\epsilon$ and for a constant $C_\epsilon \sim \frac{1}{\epsilon}$, has a unique limiting process h independently to the mollifier. We can expect that the similar result holds if $\gamma < \frac{1}{2}$ because of the local subcriticality of the equation. However, we have the following result only in $0 \leq \gamma < \frac{1}{4}$.

Theorem 0.1. *Let $\rho \in C_0^\infty(\mathbb{R}^2)$ be a smooth, symmetric, and compactly supported function integrating to 1. If $0 \leq \gamma < \frac{1}{4}$, then there exists a constant C_ϵ such that, for any initial condition $h_0 \in \mathcal{C}^\alpha(\mathbb{T})$ $(0 < \alpha < \frac{1}{2} - \gamma)$, the solutions to the equation*

$$\partial_t h_\epsilon(t,x) = \partial_x^2 h_\epsilon(t,x) + (\partial_x h_\epsilon(t,x))^2 - C_\epsilon + \partial_x^\gamma \xi_\epsilon(t,x)$$

up to some cut-off $||h_\epsilon(t,\cdot)||_{C^\alpha(\mathbb{T})} \leq L$ converges to some function h, independently of the choice of ρ. Furthermore, $C_\epsilon = \mathcal{O}(\epsilon^{-1-2\gamma})$, and the proportionality constant depends on ρ.

I appreciate that Hairer pointed out that $\gamma = \frac{1}{4}$ is a border.