# Elliptic Determinantal Processes

Makoto Katori
(Chuo University, Tokyo, katori@phys.chuo-u.ac.jp)

We introduce an elliptic extension of Dyson's Brownian motion model, which is a temporally inhomogeneous diffusion process of noncolliding particles defined on a circle with radius $r > 0$, $\mathrm{S}^1(r)$. With parameters $N \in \mathbb{N}, \alpha > 0$, and $0 < t_* < \infty$, we introduce the following function of $(t, x) \in [0, t_*) \times \mathbb{R}$,

$$A_N^{\alpha}(t_* - t, x) = \frac{1}{\alpha} \left[ \frac{d}{dv} \log \vartheta_1(v; \tau) \right]_{v=x/\alpha, \tau=2\pi i N(t_*-t)/\alpha^2},$$

where $\vartheta_1(v;\tau) = i \sum_{n\in\mathbb{Z}} (-1)^n q^{(n-(1/2))^2} z^{2n-1}$ with $i = \sqrt{-1}, z = e^{\pi i v}, q = e^{\pi i \tau}$ (the Jacobi theta function). As a function of $x \in \mathbb{R}$, it is odd and periodic with period $\alpha$. It has only simple poles at $x = m\alpha, m \in \mathbb{Z}$, and simple zeroes at $x = (m + 1/2)\alpha, m \in \mathbb{Z}$. Let $\check{\boldsymbol{X}}^A(t) = (\check{X}_1^A(t), \dots, \check{X}_N^A(t)), t \in [0, t_*)$ be a solution of the following set of SDEs on $\mathbb{R}$,

$$d\check{X}_j^A(t) = dB_j(t) + \sum_{1\le k\le N, k\ne j} A_N^{2\pi r}(t_* - t, \check{X}_j^A(t) - \check{X}_k^A(t))dt + A_N^{2\pi r}(t_* - t, \overline{X}_\delta^A(t))dt,$$

$1 \le j \le N, t \in [0, t_*)$, where $B_j, 1 \le j \le N$ are independent Brownian motions on $\mathbb{R}$, and $\overline{X}_\delta^A(t) = \delta + \sum_{j=1}^N \check{X}_j^A(t)$ with $\delta \in \pi r \mathbb{Z}$. We then define the process $\boldsymbol{X}^A(t) = (X_1^A(t), \dots, X_N^A(t)) \in [0, 2\pi r)^N, t \in [0, t_*)$ by $X_j^A(t) = \check{X}_j^A(t) \bmod 2\pi r, 1 \le j \le N$. It represents a Markov process showing the positions of $N$ particles on the circumference $[0, 2\pi r)$ of $\mathrm{S}^1(r)$.

Using elliptic determinant evaluations related to the reduced affine root system of type $A$, we give determinantal martingale representation (DMR) for the process, when it is started at the configuration with equidistant spacing on the circle. DMR proves that the process is determinantal and the spatio-temporal correlation kernel is obtained. By taking temporally homogeneous limits of the present elliptic determinantal process, trigonometric and hyperbolic versions of noncolliding diffusion processes are studied. An infinite particle limit will be discussed. Elliptic determinantal processes related to other affine root systems will be also considered.

Connection between these elliptic determinantal processes and probabilistic discrete models with elliptic weights recently studied by Borodin, Gorin, and Rains will be an interesting future problem. We note that the function $A_N^{\alpha}(t_* - t, z)$ can be regarded as Villat's kernel for an annulus $\mathbb{A}_q = \{z \in \mathbb{C} : q < |z| < 1\}$ with $0 < q = e^{-2\pi^2 N(t_*-t)/\alpha^2} < 1$. It is the reason why it also appears in the study of stochastic Komatu-Loewner evolution in doubly connected domains

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